A simple experiment for studying the transition from order to chaos

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(Received 26 April 1985; accepted for publication 24 September 1985)

A simple experiment is described for the observation of transition from order to chaos in a near Hamiltonian and a strongly dissipative system. The system is a compass needle moving in an oscillating magnetic field. As the field amplitude is increased, period doubling is observed following the Feigenbaum scenario. New modes are also created by tangent bifurcations. The motion in the dissipative case appears to terminate in strange attractor-type chaotic motion.

I. INTRODUCTION

Routes from order to chaos have recently attracted much attention. Feigenbaum1 analyzed one-dimensional maps and found that a large variety of these maps undergo period doubling bifurcations leading to chaos that are universal in some important aspects. Other authors have shown2 that Hamiltonian systems have their own universal bifurcation sequences. Here we wish to describe an experiment on a simple dynamical system that shows this transition in the nearly Hamiltonian as well as the strongly dissipative case.

Consider a compass needle in a spatially uniform magnetic field that oscillates in time. If the angle between the compass needle of dipole moment $m$ and the magnetic field direction is $\theta$, the torque exerted on the dipole is $mB \sin \theta$, where $B = B_0 \sin \omega t$ is the time varying magnetic field. This leads to the equation of motion

$$ \dot{\theta} = mB \sin \omega t \sin \theta, $$

(1)

where $I$ is the moment of inertia of the dipole. This equation despite its apparent simplicity is nonintegrable (just like the three body problem) and has been studied by several authors.3–5 Experiments on this system have also been reported.6

This mechanical system has a Hamiltonian

$$ H(\theta, p, t) = \frac{p^2}{2I} + mB_0 \cos \theta \sin \omega t. $$

(2)

The phase space is three dimensional with $\theta, p = I \dot{\theta}$, and $t$ the coordinates. The system is evidently periodic in $t$, and it is convenient to visualize time as a toroidal coordinate in phase space.

Consider now the surface of sections, the $\theta, p$ plane at times when $\omega t = 2\pi n$, $n$ integer. The phase-space trajectory intersects this plane in a sequence of points. This sequence of points corresponds to a mapping on the plane that is known to be area preserving. Some orbits may give fixed points, e.g., points mapping into themselves, or periodic orbits that return to their initial $\theta, p$ point after an integral number of iterations.

If this system follows the universal scenario, then increasing a parameter (like $B_0$) gradually will cause a fixed point to bifurcate by period doubling, e.g., it becomes a periodic orbit of period 2, further increase in $B_0$ results in a bifurcation to period 4 and so on to period $2^k$. When $n \rightarrow \infty$ the system has become chaotic. Furthermore if one designates $A_k$ the value of $B_0$ where the $k$th bifurcation takes place, then the ratio

$$ \lim_{k \rightarrow \infty} (A_k - A_{k-1})/(A_{k+1} - A_k) = \delta $$

is a universal constant. There are other universal constants $\alpha$ and $\beta$ associated with the process.2

In the case of our compass needle, one may think of the oscillating magnetic field as being composed of two counter-rotating fields:

$$ \dot{\theta} = \left( mB_0/2I \right) \left[ \cos(\omega t - \theta) - \cos(\omega t + \theta) \right]. $$

(3)

The equation of motion in either one of the rotating field components is integrable, and may result, e.g., in the corotation of the needle with this field component. It is the interaction with both components simultaneously that renders the system nonintegrable. As long as $B_0$ is sufficiently small, $\theta = \pm \omega t + \pi/2$ is a good approximate solution.

A good way to estimate when $B_0$ is "small" in this context can be obtained as follows. Considering one rotating field component at a time one finds

$$ \dot{\theta} = \left( a/2 \right) \cos(\omega t + \theta), $$

(4)

where $a = mB_0/I$. With the new variable $\gamma = \theta + \omega t$ this is just the pendulum equation that can be integrated without difficulty. In $\theta, \dot{\theta}$ phase space there is a separatrix with half-width $\delta\dot{\theta} = \sqrt{a}$. An orbit inside the separatrix corresponds to corotation with the field component and superposed oscillation. As long as the separatrices for the two field components are far removed, a corotating orbit is only slightly disturbed by the other field component7 (Fig. 1). This leads to the condition $\sqrt{a} < \omega$ or $B_0 < (I/m)\omega^2$. Manifestly nonintegrable behavior, period doubling bifurcations, and chaos can be expected when this condition is violated.

The surface of section plot can be conveniently generated by flashing a stroboscopic light on the magnet needle, synchronized to the oscillating field. When the dipole corotates with one of the field components (e.g., rotating clockwise) the flash will always find the needle in the same position (having rotated $2\pi$ during the dark period), and one concludes that there is a stable fixed point on the surface of sections. As $B_0$ is increased one expects a bifurcation to take place and the needle will alternate between two positions at subsequent flashes. Further increase in $B_0$ results in period 4 and so on. At the end of this bifurcation sequence the motion becomes chaotic. In deterministic chaos the system exhibits sensitive dependence on initial conditions; a small change in initial conditions results in a trajectory which deviates from the original exponentially with time. Unless one knows all previous positions of the needle with infinite precision, its position at the next flash is as unpredictable as the next outcome of a coin toss.

The Hamiltonian system described by Eq. (1) is strictly unattainable in the laboratory since dissipation is always present. The dissipative system is also of great interest since the corresponding value of $\delta$ is different from that found in
the Hamiltonian case. In order to produce a strongly dissipative system the compass needle was equipped with a damping tube immersed in a viscous liquid. The appropriate equation is then

\[ \dot{\theta} = A \sin 2\pi \theta \sin 2\pi t - B \dot{\theta}, \]  

(5)

where the normalization \( \omega t \to 2\pi t \) and \( \theta \to 2\pi \theta \) has been adopted, and \( A = 2\pi a/\omega^2 = 2\pi mB_0/\omega^2 \).

It should be stressed that as long as the system follows the universal period doubling bifurcation sequence, the exact form of the dissipative term is not important.

We have carried out experiments with both, the nearly Hamiltonian and the strongly dissipative system. The first few bifurcations and the ensuing apparent chaos are clearly visible. Higher bifurcations are difficult to measure, due to the extreme sensitivity of the system to external noise (shaking of the building, stray fields, etc.).

Beyond the bifurcation sequence the chaotic motion appears to be describable in terms of strange attractors. Such motion has also been observed in this experiment.

II. EXPERIMENTAL SETUP

The system consists of a magnet needle in a protective cover with graduations underneath. One end of the needle is painted red, the other white.

For low damping a commercial air-filled compass was used. For heavier damping a water-filled compass was tried. However, when the needle rotated through the water it shed vortices, another nonlinear effect. No regular sequence of period doubling could be found.

The water damping was then replaced by a “Couette”-type system: The magnet needle was equipped with a \( \frac{1}{16} \) in. thin-walled tube, concentric with its Agate bearing. The mount was modified to provide a cylindrical well, \( \frac{1}{4} \) in. in diameter, concentric with the needle supporting the magnet needle. The \( \frac{1}{16} \) in. tube dipped into this well. The well was filled with Dow Corning 200 silicone oil with a viscosity of either 1000 centistokes (cS) or of 12500 cS. The depth of immersion could be varied between \( \frac{1}{16} \) in. and \( \frac{3}{16} \) in., providing further adjustment of the damping. This system was still not completely free of flaws: The free surface of the oil can sustain waves, which may be bothersome because this silicon oil is a non-Newtonian fluid.

The magnet needle sits in the center of a large set of Helmholtz coils compensating the Earth’s magnetic field. A smaller set of Helmholtz coils is nested inside the first set, providing a spatially uniform magnetic field varying sinusoidally in time.

The sine wave is generated by a HP Model 3300 signal generator and fed through a precision attenuator to an emitter-follower power amplifier. This amplifier can handle voltage swings of \( \pm 30 \) V and currents up to \( 1 \) A.

The frequency is measured with a counter. The positive and negative current swings are measured with a 1-\( \Omega \) standard resistor, two peak detectors, and a HP 3440A digital voltmeter.

The signal generator also provides a synchronous square wave which was used to trigger the stroboscope either directly or over an inverter and/or delay circuit.

The parameter \( A \) in Eq. (5) is determined by timing oscillations of the magnet needle in a constant magnetic field, \( B_{dc} \), supplied by the same Helmholtz coils used for the ac field.

For small angles, and negligible damping, the frequency of oscillations is

\[ f_0 = \frac{1}{2\pi} \sqrt{mB_{dc}/I}. \]

The damping constant \( B \) was obtained by observing the magnet needle while rotating at smaller and smaller amplitudes of the sinusoidally varying magnetic field. Clearly, with damping present, a finite field amplitude is required to sustain rotation.

It can be shown, that for small damping, the smallest value of \( A, A_{\text{min}} \), which sustains rotation is given by

\[ A_{\text{min}} = 2B. \]

For \( A \) slightly larger than \( A_{\text{min}} \), the rotation is almost uniform but phase shifted by an angle \( \gamma \) (in radians):

\[ \theta = t + \gamma. \]

The phase shift \( \gamma \) depends on \( A \) as

\[ \cos \gamma = \frac{2B}{A}. \]

At \( A_{\text{min}} \) \( \gamma \) is zero, at larger values of \( A \) \( \gamma \) approaches \( \pm \pi \), the sign depending on the sense of rotation.

As \( A \) is increased, the originally almost uniform rotation becomes nonuniform. It does, however, repeat itself in the first and second half-cycle: The repetition time is \( T/2 \), where \( T \) is the period of the sine wave.

As \( A \) is increased further, the second half-cycle no longer repeats the first half-cycle: A bifurcation from \( T/2 \) to \( T \) has taken place.

This first bifurcation is observed by triggering twice each cycle, \( 180^\circ \) apart, at the top and the bottom of the sinewave. Originally, below the bifurcation, the red and the white end appeared at the same spot for the two flashes. As the value of \( A \) exceeds a value \( A_1 \), they appear first at neighboring and then at wider and wider removed points.

The stroboscope is now triggered once each cycle and the field, that is \( A \), is increased further. When a value \( A_2 \) is exceeded, the position of the white end is at two different places at alternate flashes: The repetition length is \( 2T \).
III. EXPERIMENTAL RESULTS FOR SMALL DISSIPATION

A typical experimental result for the near-Hamiltonian system is shown on Fig. 2. It is clear that $A_{\text{min}}$ is very small compared to the first bifurcation point $A_1$, indicating near-Hamiltonian behavior. Two bifurcations ($T/2 \rightarrow T$ and $T \rightarrow 2T$) are clearly visible. In our normalized Eq. (5) $T = 1$. No bifurcations beyond period 2 orbits have been observed. The bifurcation point $A_3$ ($2 \rightarrow 4$) is very close to $A_\infty$, the infinite period limit of period doubling bifurcations. As pointed out in the introduction the distances between the $A_k$ values shorten as a geometric series with a characteristic factor $\delta$. For Hamiltonian systems $\delta = 8.721...$

Apparent chaos is found to the right of the last $A$ value in Fig. 2. The position of the magnet needle appears to be random (unpredictable) at successive stroboscopic plots. Furthermore the needle no longer has a distinct direction of rotation. For a while it rotates to the right, then to the left with no apparent regularity. The transition from bifurcated order to chaos is sudden and dramatic.

Our method measures the angle $\theta$, but the representation on the surface of sections requires the value of $\phi$ as well.

This can be obtained by gradual phase shifting of the stroboscopic flashes to obtain $\theta$ as a function of time. The result of such measurements is shown on Fig. 3. For small $A$ one has nearly uniform rotation resulting in a nearly linear $\phi(t)$ curve, as the dipole is driven by the field component rotating in one direction (say clockwise) and is nearly unaffected by the other rotating field component. As $A$ is further increased the other field component influences the motion more and more. In the language of Fig. 1 the separatrices move closer and actually overlap. It is easy to see that separatrix overlap begins when $A = 2\pi$. On Fig. 3 when $A = 9.252$, during one period, the rotation is first clockwise, then stops and becomes counterclockwise for a short time, then stops and becomes clockwise again. This is still regular, but bifurcated motion.

A surface of section plot for period 1 and 2 orbits is shown on Fig. 4. The location of the cycle elements depends on course on the time chosen in the oscillation period of the field. In the figure $t_1 = 0$ and $t_2 = 0.5$ were chosen.

In a Hamiltonian system one expects the existence of an infinity of other stable periodic orbits. Figure 2 also shows a period 3 orbit that has been observed. (This mode has apparently been created by a tangent bifurcation from the fundamental mode).

In addition to the rotating modes, other types of motion can also be observed. In the experiments described so far, the dipole was initially given a left or right rotation by a permanent magnet rotated over the dipole and then withdrawn. If however, the dipole is initially aligned parallel to the symmetry axis of the coils, and slightly perturbed, damped oscillations are observed for small $A$. As $A$ is increased beyond a critical value $A_c$, steady oscillations are observed. $A_c$ can be calculated analytically by expanding Eq. (5) for small $\theta$ and neglecting $B$, to obtain

$$\dot{\phi} = 2\pi A \sin 2\pi t \phi.$$ (6)

This is a Mathieu equation (without the constant $\theta$ term) whose stability properties are known. As $A$ is increased beyond $A_c = 2.853$ unstable growing oscillations ensue. The value of $A_c$ is in good agreement with the experimental value.

In our experiment the growing oscillations predicted by Eq. (6), terminate in a limit cycle of regular oscillations with period 2. This is not unexpected since Eq. (6) holds only in the $\phi \rightarrow 0$ limit. For $A > A_2$ the oscillation amplitude grows as $A$ increases and terminates in chaotic motion. Numerical solutions of Eq. (5) show a period doubling bifurcation sequence to chaos, not observed in our experiment.

IV. EXPERIMENTS WITH STRONG DISSIPATION

The next set of experiments was performed with a damping tube with silicon oil, that produces variable damping.
Fig. 5. Experimental results for heavily damped magnet needle, rotating motion.

One expects that for strong damping the Feigenbaum ratio approaches \( \delta_D = 4.469 \).

(To be more precise, \( \delta \) is defined as a limit \( k \to \infty \), and it may be shown\(^1\) that this limit should approach 4.469 no matter how small the dissipation. Experimentally however this limit is unattainable since one can look only at the first few bifurcations. For those one expects \( \delta_H = 8.7 \) for the near Hamiltonian and \( \delta_D = 4.47 \) for the strongly dissipative case.\(^1\)

With the effective \( \delta \) thus reduced one expects to see more bifurcations. In fact the experimental results shown in Fig. 5 clearly exhibit transitions 1\( \to \)1, 1\( \to \)2, and 2\( \to \)4. This is sufficient to calculate an experimental approximation to \( \delta \).

The oscillating mode, described for the near Hamiltonian system also shows one more bifurcation (2\( \to \)4) as shown in Fig. 6.

In the mathematical literature much significance is attached to period 3 modes generated for very large driving parameters beyond chaotic regions; "Period three implies chaos."\(^5\) We have in fact observed an oscillating mode for \( A = 42.04 \), whose period is three times the period of oscillation of the field. Figure 7 shows the angle \( \theta \) as a function of time for this mode. Notice that the damping \( B = 6.95 \) is very large, so comparison with one dimensional maps is not unreasonable.

The chaotic region for appreciable damping is characterized by strange attractors.\(^6\). The surface of section plots of these attractors, while chaotic, do not cover all available phase space, but only a line with an infinity of folds. One expects that the orbits of the magnet needle for \( B \neq 0 \), past \( A_w \), show strange attractor-type behavior. The transition for \( B \to 0 \) is an interesting question to be described elsewhere.\(^7\)

We have found an orbit that we believe belongs to a strange attractor. As \( A \) is increased in Fig. 5 to 12.8 the motion "slips phase" and goes to a different rotating mode with an apparent period 1. It was observed, however, that this motion is only quasiperiodic and successive measurements show an appreciable scatter in the position of the needle at successive flashes. To rule out experimental error we have reproduced this mode on the computer. The scatter of points, observed in the experiment is in fact reproduced by computer calculations. The orbit appears to be that of a two-piece strange attractor (Fig. 8).

Fig. 6. Experimental results for heavily damped magnet needle, oscillations around direction of magnetic field.

Fig. 7. Experimental results for very heavily damped magnet needle, oscillations with period 3. Note that the oscillation is not centered around field direction.

Fig. 8. Surface of section plot. Computer data for a two-piece strange attractor.

\(^{10}\)\(^{11}\)\(^{12}\)
V. SUMMARY

We described a simple experiment, suitable for student laboratory use, on the transition from order to chaos. Period doubling bifurcations and chaotic motion are clearly visible in both the near Hamiltonian as well as the dissipative case. The generation of new modes by tangent bifurcations can also be observed. The chaotic trajectories observed in the strongly dissipative case appear to be strange attractors, as confirmed by numerical integration of the differential equation describing the system.

ACKNOWLEDGMENTS

Over the years that this experiment was built and data taken, several students participated in the project. We wish to thank I. Sezan and E. Gottschalk for their contributions. The work received partial support from DOE No. DE-

An alternate derivation of relativistic momentum

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(Received 5 August 1985; accepted for publication 13 September 1985)

An alternate derivation of the expression for relativistic momentum is given which does not rely on the symmetric glancing collision first introduced by Lewis and Tolman in 1909 and used by most authors today. The collision in the alternate derivation involves a non-head-on elastic collision of one body with an identical one initially at rest, in which the two bodies after the collision move symmetrically with respect to the initial axis of the collision. Newtonian momentum is found not to be conserved in this collision and the expression for relativistic momentum emerges when momentum conservation is imposed. In addition, kinetic energy conservation can be verified in the collision. Alternatively, the collision can be used to derive the expression for relativistic kinetic energy without resorting to a work-energy calculation. Some consequences of a totally inelastic collision between these two bodies are also explored.

I. INTRODUCTION

The concept of relativistic momentum and its relation to relativistic mass $p = mL = \gamma m_0$ are at the foundation of relativistic dynamics. In most elementary treatments of relativity, the expression for relativistic momentum is found by imposing momentum conservation on a particular elastic collision of two identical bodies. In this collision viewed in one frame, one body makes a symmetric, glancing collision with another body that only moves in the $+y$ direction. The same collision is then viewed in another frame, in which the first body only moves in the $\pm y$ direction and the second makes the symmetric, glancing collision. Symmetry of the problem in the two frames, transformation of components of velocity, and the limit of a very slight collision then lead to the correct expression for relativistic momentum.

In a field such as special relativity, in which there are usually many examples and paradoxes to illustrate particular points, it is rather strange to find only one example for finding the expression for relativistic momentum from momentum conservation. Perhaps this would not be surprising if the collision used were one which is natural or encountered often. But the fact is that this particular collision is an artificial one, cooked up solely to find the expression for relativistic momentum, and not used in subsequent discussions. Moreover, in the usual treatment of the problem, the expression for the relativistic momentum (or relativistic mass) emerges only in the limit of zero angle of deflection of the collision, i.e., in the limit in which no collision takes place. These features lead one to ask if there might not be another way to find the expression for relativistic momentum. One possible alternative is the subject of this paper.

II. AN ELASTIC COLLISION

Consider the following elastic collision between two identical bodies. In the $S$ frame before the collision, one body has an initial velocity $u_i$ in the $+x$ direction and the