Energy conservation and chaos in the gravitationally driven Fermi oscillator

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Chaotic dynamics of the two-body conservative system, which consists of a particle of mass \( m \) bouncing elastically on a horizontal plate of mass \( M \) supported by an elastic spring, is investigated. The system integrates the properties of coupled oscillators with those of the bouncing-ball and impact oscillator problems. Results obtained by varying the mass ratio \( M/m \) and the spring constant \( k \) in numerical computations are presented in the form of time-dependent diagrams and discrete maps. The rich variety of resulting chaotic behavior includes strange attractors with fractal structure, resonant islands, crisis, and intermittency route to chaos. The system has a remarkable didactic value as an example of chaotic behavior in simple systems close to everyday experience. The integrable limit \( M=0 \) is appropriate for introducing the phase portraits and discussing the interrelationship between the shape of the potential energy curves and the resulting oscillatory motion. © 1998 American Association of Physics Teachers.

I. INTRODUCTION

Exploring possible mechanisms for acceleration of cosmic particles trapped in the Earth’s magnetic field, in 1949 Fermi\(^1\) proposed the system consisting of a ball bouncing between one fixed and one oscillating wall [Fig. 1(a)]. The resulting one-particle motion is referred to as the Fermi acceleration, and a broad class of related problems are known as Fermi oscillators.\(^2\)\(^-\)\(^5\)

In this paper we consider the conservative two-body system which is essentially a gravitationally driven Fermi oscillator.\(^2\) It consists of a particle and a horizontal plate supported by an elastic spring [Fig. 1(b)]. When dropped on the plate, the particle bounces elastically, exhibiting vertical free fall motion until the next collision with the plate. At the same time the plate is accelerated to oscillate harmonically. The resulting motion is highly sensitive to the ratio of the plate mass \( M \) and the particle mass \( m \).

Newtonian equations of motion are solved analytically in the time interval between the two collisions. Exact solutions, however, depend on the initial positions and velocities, phases and amplitudes at the moment of impact. These initial conditions not only differ from one collision to the next, but represent apparently disordered sequences of data. We are confronted with a typical problem of classical chaotic dynamics in one geometrical dimension developed within a simple conservative two-body system.
The motions of the ball and the plate are consequences of the energy transfer between the two bodies during the collision, and, separately for each of them, of the continuous transformations between the kinetic energy, elastic potential energy, and gravitational potential energy during the intervals between the two collisions. Problems of this type occur in many realistic situations. They are found in engineering applications and practical appliances, they underlie the functioning of toys and experimental setups, and are met in many sports, governing, for example, the impact between the ball and the racket in tennis. In some of its aspects this problem is related to the bouncing ball and impact oscillator problems with dissipation, where, however, only a very massive wall vibrating with constant amplitudes was considered. On the other hand, some properties of the case presented here remind one of systems with two coupled oscillators.

In the present paper we investigate the dynamics of this plate–ball system. Before going into details, we wish to stress the great didactic value of the analyzed problem. It is fascinating that one finds such an exemplary chaotic system only a short step away from the usual subjects of introductory university physics. The example presented here has some additional advantages. With the limit of the zero plate mass included, it also encompasses the integrable one-body conservative behavior, suggesting the method for introducing into the physics curriculum the phase portraits and the potential energy curves. In this way the difficult concept of energy conservation is approached from a different angle, with special emphasis on the importance of the shape of the potential energy curves for the resulting oscillatory motion. Varying \( M/m \) as the control parameter, one is able to describe and illustrate the fundamental chaotic phenomena such as attractors or time series on the basis of a system close to realistic everyday experience.

In Sec. II we briefly describe the variables and parameters of the investigated system and fix the notation which will be used in the text. In Sec. III we describe the motion of the particle of mass \( m \) when the plate mass is equal to zero \((M=0)\). The obtained results are used in a discussion of the relation between the potential energy dependence and the oscillatory behavior. In Sec. IV the case with \( M \neq 0 \) is investigated. Equations of motion are exactly solved and consequences of the impacts are calculated. This approach yields both the continuous time dependence and the discrete maps for the evolution of the system. Poincaré sections for particle position–particle velocity at the moment of impact are presented for a large number of successive impacts. The concepts of time series, intermittency, and chaotic attractor are introduced and illustrated with numerical results. In Sec. V we summarize the ideas and results of our investigation, and conclude by discussing the educational and didactical values of the described dynamic system.

II. DESCRIPTION OF THE SYSTEM

The system in which a particle bounces on the plate supported by an elastic spring is schematically shown in Fig. 1(b). The following parameters determine its dynamics: the mass \( m \) of the particle, the mass \( M \) of the plate, the elastic spring constant \( k \), the gravitational acceleration at the Earth’s surface \( g \), and the total energy \( E \). In our calculations \( E, g, \) and \( m \) will be held fixed, and the behavior will depend essentially on two parameters, plate mass \( M \) and the spring constant \( k \). It is convenient to introduce several new symbols: the plate oscillation frequency \( \omega \), the initial height \( h \) of the particle, and the gravitational free fall time \( T_{gr} \) for the particle:

\[
\omega = \sqrt{\frac{k}{M}}, \quad h = \frac{E}{mg}, \quad T_{gr} = \sqrt{\frac{2h}{g}}.
\]

We also make use of dimensionless parameters \( \mu, \kappa, \) and \( \Omega \):

\[
\mu = \frac{M}{m}, \quad \kappa = \frac{2hk}{mg}, \quad \Omega = \omega T_{gr}.
\]

In the calculations presented in this paper it will be assumed that the plate is initially at rest. The particle is dropped from the height \( h \) and reaches the plate at the moment \( t=0 \) with the velocity \( v_0 = -\sqrt{2gh} \). The following constant values were taken:

\[
m = 1 \text{ kg} \quad \text{for the particle mass} \]
\[
h = 4.905 \text{ m} \quad \text{for the initial height}. \]
\[
g = 9.81 \text{ m/s}^2 \quad \text{for the gravitational acceleration}.
\]

This is equivalent to the total energy of \( E = mgh = 48.118 \text{ J} \) and to \( T_{gr} = 1 \text{ s} \).

III. MOTION OF THE PARTICLE BOUNCING ON THE MASSLESS PLATE

A. Equations of motion

Here we consider the special limiting case with zero plate mass: \( M = 0 \) or \( \mu = 0 \). The particle of mass \( m \) moves vertically in the gravitational field, bouncing on the massless plate supported by a spring of elastic constant \( k \). This system is one dimensional and conservative and its motion is regular. The particle falls from height \( h \) and reaches the plate at time \( t = 0 \). The total energy of the particle is
B. Results and discussion

The equations of motion for the two regions of $x$ are

$$E_p(x) = \begin{cases} mgx + \frac{1}{2}kx^2 & \text{if } x \leq 0 \\ mgx & \text{if } x > 0. \end{cases}$$

The phase portraits follow from (1), (4), and (5) giving

$$v(x) = \begin{cases} \pm \sqrt{2gh} \left( -1 + \frac{1}{\kappa} \sin \left( \sqrt{\frac{t-t_*}{T_{gr}}} \arcsin \left( \frac{1}{1+\kappa} \right) \right) \right) & \text{if } x \leq 0 \\ \pm \sqrt{2gh} \left( t-t_* \frac{1}{2} \left( \frac{t-t_*}{T_{gr}} \right)^2 \right) & \text{if } x > 0. \end{cases}$$

Here the beginning of the $n$th interval in which this motion happens is given as

$$t_n = nT \quad (nT \leq t < nT + 2T_{vib}) \quad \text{if } x \leq 0,$$

$$t'_n = nT + 2T_{vib} \quad (nT + 2T_{vib} \leq t < nT + T) \quad \text{if } x > 0,$$

where $T$ is the period, $T_{gr}$ is given by (1), and

$$T_{vib} = T_{gr} \frac{1}{\sqrt{\kappa}} \left( \arcsin \left( \frac{1}{1+\kappa} \right) + \frac{\pi}{2} \right).$$

The motion described by (8) and (9) consists of two parts. At first the particle moves harmonically, sticking to the plate. It reaches the equilibrium point

$$x_{eq} = \frac{-2h}{\kappa},$$

and then returns, exhibiting harmonic oscillation during the time $2T_{vib}$. After reaching the initial position, it leaves the plate and goes up and down during the time $2T_{gr}$ following the law of free fall. Hence, the period is

$$T = 2T_{gr} + 2T_{vib}.$$
This is reflected also in the phase portraits, plotted from Fig. 4. The two extremes in Fig. 4 are examples of transformations between kinetic energy, gravitational energy, or combined elastic and gravitational in the transitional cases. Phase portraits for both pure and transitional cases could be an interesting, novel and enlightening method to analyze the motion. Therefore they deserve to be introduced into textbooks and curricula, together with the more traditional time-dependence approach.

IV. PARTICLE BOUNCING ON A PLATE WITH MASS $M \neq 0$

A. Equations of motion

In this section we investigate cases with $M \neq 0$ $(\mu \neq 0)$. The total energy of the system is

$$E = \frac{mv_x^2}{2} + \frac{Mv_y^2}{2} + mgx + Mg y + \frac{1}{2} k y^2. \quad (16)$$

The system has two degrees of freedom. The phase space is spanned by four variables: position of the particle $x$, position of the plate $y$, velocity of the particle $v_x$, and the velocity of the plate $v_y$. The equations of motion are

$$m\ddot{x} = -mg, \quad M\ddot{y} = -Mg - ky. \quad (17)$$

B. Results and discussion

After being launched with the initial positions and velocities, the particle and the plate are subject to a series of elastic impacts. For the time intervals between two collisions, equations (17) are solved analytically. If the particle and the plate positions at the moment of the $n$th impact are $x_n = y_n$ with the departing velocity of the particle $v_{x,n}$ and the departing velocity of the plate $v_{y,n}$, in the interval between the $n$th and $n + 1$th impact the positions depend on time as follows:

$$x = x_n + v_{x,n}(t-t_n) - \frac{1}{2}g(t-t_n)^2, \quad (18)$$

$$y = -\frac{g}{\omega_n^2} + A_n \sin[s_n\omega(t-t_n) + r_n \arctan Q_n], \quad (19)$$

with

$$Q_n = \left[ \frac{v_{y,n}'}{\omega_n} \right] \left[ y_n + \frac{g}{\omega_n^2} \right]. \quad (20)$$

The corresponding velocities are

$$v_x = v_{x,n} - g(t-t_n), \quad (21)$$

$$v_y = A_n s_n \cos[s_n \omega(t-t_n) + r_n \arctan Q_n]. \quad (22)$$

The plate oscillation amplitude between the two impacts is

$$A_n = \sqrt{\left( \frac{v_{y,n}'}{\omega_n} \right)^2 + \left( y_n + \frac{g}{\omega_n^2} \right)^2} \quad (23)$$

and the sign factors in (19) and (22) are

$$s_n = \text{sign}(v_{y,n}'), \quad r_n = \text{sign}(y_n + \frac{g}{\omega_n^2}). \quad (24)$$
The time \( t_{n+1} \) of the \( n \)th impact is obtained by finding the intersection of functions (18) and (19). The velocities calculated for this time are the approaching velocities for the \( n+1 \)th impact \( v_{x,n+1} \) and \( v_{y,n+1} \) of the particle and the plate, respectively. The departing velocities \( v'_{x,n+1} \) and \( v'_{y,n+1} \) follow from the conservations of momentum and energy in the collision. The resulting seven equations,

\[
\begin{align*}
x_{n+1} &= x_n + v'_{x,n}(t_{n+1} - t_n) - \frac{1}{2g}(t_{n+1} - t_n)^2, \\
y_{n+1} &= -\frac{g}{\omega^2} + A_n \sin[s_n \omega (t_{n+1} - t_n) + r_n \arctan Q_n], \\
y_n &= x_n + 1, \\
v_{x,n+1} &= v'_{x,n} - g(t_{n+1} - t_n), \\
v_{y,n+1} &= A_n s_n \cos[s_n \omega (t_{n+1} - t_n) + r_n \arctan Q_n], \\
v'_{x,n} &= \frac{2\mu v_{x,n} + (1 - \mu)v_{x,n}}{1 + \mu}, \\
v'_{y,n} &= \frac{2v_{x,n} + (\mu - 1)v_{y,n}}{1 + \mu},
\end{align*}
\]

(25)–(31) determine exactly the particle and plate positions as well as the approaching and departing velocities for the \( n+1 \)th impact, as functions of the known positions and velocities for the \( n \)th impact. The starting data consistent with (3) are

\[
\begin{align*}
x_0 &= 0, \quad v_{x,0} = -\sqrt{2gh}, \\
y_0 &= 0, \quad v_{y,0} = 0.
\end{align*}
\]

Equations (23)–(31) yield the sequences of data (positions, velocities, amplitudes, impact times, etc.) which define discrete maps and allow the geometrical representation of the dynamics of the given problem in the form of the Poincaré diagrams. The definition of the map given by these equations seems to be more complicated than are the usual simple difference equations. However, it has to be born in mind that the map defined by (23)–(31) is the exact solution of the defined Fermi oscillator problem, taking care of the signs and the variable amplitudes. Besides, it has been known that for the impact oscillators, even the evaluation of apparently simple maps leads to numerical difficulties and is extremely sensitive to details of the computational techniques.

In our calculations the basic computational problem is to find the intersection (27) of (25) and (26). We have written a computer code based on the interval bisection method, which converges very rapidly. Since we were dealing with a large number of successive impacts (up to \( 5 \times 10^5 \)), the precision of the calculated results was treated with utmost care. Control mechanisms giving an indication of the possible errors had to be built into the program. First of all, energy conservation was verified throughout the calculation, and the parameter \( \sigma = (E - E_0)/(E + E_0) \) (where \( E_0 \) the total energy at the beginning of the iteration and \( E \) the ending value) was smaller than \( 10^{-11} \). Another important condition has to be fulfilled: The numerical algorithm must assure that no intersection points is missed. It is important because this kind of error can lead to false quasiperiodic solutions and may affect the interpretation of results. In writing and applying our code, this point has been given due attention.

Our calculations can be divided into two groups. First are the discrete series of data connected with the successive impacts. In these calculations the number of impacts was very large and most figures shown in this paper were calculated with \( 2 \times 10^5 \) impacts. The second type of results describes the time evolution of the system—its positions and velocities. There the number of impacts was smaller, but when considered necessary, it could be appreciably extended.

Preliminary calculations have shown that by varying the mass and spring constants one obtains an extremely rich palette of dynamical patterns. Here we present calculations of the dependence of particle and plate positions on time and, in parallel with them, the Poincaré maps connecting the position of the particle and its approaching velocity for a sequence of successive impacts. The results are shown in Figs. 5–8 and are organized as follows. In Fig. 5 the plate oscillating frequency is held constant at the value \( \omega = 6 \text{ s}^{-1} (\Omega = 6) \). The mass ratio \( \mu = M/m \) is decreased from 1000 to 1. In Fig. 6 the same plate frequency is taken, but the mass ratio is equal to 1 or smaller. In Fig. 7 the mass ratio is varied from 1 to 100, whereas the spring parameter is held constant (\( k = 100 \text{ N m}^{-1} \) or \( k = 1000 \)). Figure 8 presents the corresponding results for \( \kappa = 100 \) and several values of the mass ratio \( \mu < 1 \).

We shall briefly discuss the essential features of the resulting diagrams. We start with Fig. 5(b) showing a small region of the phase plane containing all calculated points for \( \Omega = 6 \) and \( \mu = 100 \). They all lie on a closed curve which is evidently a simple attractor of fractal dimension 1. When increasing \( \mu \), as for example for the case \( \mu = 1000 \) shown in Fig. 5(a), the points define a group of tiny closed curves, indicating an appreciable degree of regularity in the behavior of the system for this choice of parameters. For even larger \( \mu \), these contours shrink practically to points. On the other side, when the value of \( \mu \) is decreased from 100 to smaller values, the simple attractor is slowly transformed into an interesting strange attractor with a fractal structure and great complexity. Finally, when a certain critical value of \( \mu \) is reached, the crisis prevails and for larger \( \mu \), the chaos eventually invades the phase plane and the trajectories fill almost the whole accessible region. The time-dependence diagrams show for this region of parameters there are no great variations of amplitudes, and therefore the obtained attractor could be related to attractors found in the constant amplitude approximation to the bouncing ball problems. An analysis of Fig. 5(a)–(b), completed with our earlier discussions, suggests what we have to do here with an interesting bifurcation structure. Since the bifurcations, especially those for the impacts oscillators, are in themselves a vast and computationally intricate subject, their properties for the given system are investigated separately and will be treated elsewhere.

Among many other different chaotic situations depicted in Figs. 6–8, we point out Fig. 6(b) for \( \mu = 1/3 \) and \( \Omega = 6 \), exhibiting a structure of resonant islands typically met in conservative chaotic systems.

Figures 5–8 contain only a small part of the data obtained in the calculation. Other calculated data are shown in Figs. 9–12. First we show examples of how different data plots reflect the existence of the strange attractor in the dynamics of the systems. In Fig. 9, for three values of \( \mu \) we show the
following results: in Fig. 9(a) the plots connecting the successive time intervals between the two impacts, in Fig. 9(b) the dependence of the amplitude on the time interval, in Fig. 9(c) the dependence of the successive positions, and in Fig. 9(d) the dependence of the incoming plate velocity on position at the moment of the impact. An additional plot shown in Fig. 9(e) is devoted to phases. In Sec. IV the phases in Eqs. (26) and (29) have been expressed by means of the
signum variables (24). This was convenient from a computational point of view and was also a transparent way to keep trace of signs in the initial conditions. We can, however, use the identity

\[
\sin[\theta(t) + \phi_n] = \sin[\theta(t) + \phi_n],
\]

defining the phase \(\phi_n\) as

\[
\phi_n = s_n r \arctan Q_n + \left[1 - r_n \frac{1 + s_n}{2}\right] \pi.
\]

The last term ensures that the phase is taken modulo 2\(\pi\).

In Fig. 9(e) phase (34) (divided by \(\pi\)) at the moment of the \(n + 1\)th impact is plotted against the corresponding phase for the \(n\)th impact. Analyzing Fig. 9(a)–(e) one is impressed by seeing how the attractors typical of certain choices of parameters reflect themselves in the time series, position, phase, and amplitude diagrams. This example is therefore a good illustration of the fact that by measuring the time sequences and other data in experiments, one can discover the attractors and identify the typical features of a chaotic dynamical behavior.

Fig. 6. Dependence of the particle position (grey line) and the plate position (black line) on time (left), and the particle position–particle velocity map for the first \(2 \times 10^5\) collisions (right). The parameter \(\Omega = \omega T_g = 6\). The mass of the plate is equal to or smaller than the particle mass. The following cases are shown: (a) \(M/m = 1(\kappa = 36)\); (b) \(M/m = 1/3(\kappa = 12)\); (c) \(M/m = 1/10(\kappa = 3.6)\); (d) \(M/m = 1/50(\kappa = 0.72)\).

Fig. 7. Dependence of the particle position (grey line) and the plate position (black line) on time (left), and the particle position–particle velocity map for the first \(2 \times 10^5\) collisions (right), for \(\kappa = 100\). The mass of the plate is equal to or greater than the particle mass. The following cases are shown: (a) \(M/m = 1(\Omega = 10)\); (b) \(M/m = 25(\Omega = 2)\); (c) \(M/m = 100(\Omega = 1)\).

\[
\sin[s_n \omega(t - t_n) + r_n \arctan Q_n] = \sin[\omega(t - t_n) + \phi_n].
\]

Figure 11 shows the time dependence of the impact positions for the case with \( \mu = 100 \) and \( \kappa = 100 \) for a longer time interval (\( \approx 10^5 \) collisions). There we find an example of the intermittency in the onset of chaos, and comparison with Fig. 7(c) suggests the correlation between such intermittent chaoticity and the existence of multiple attractors.

As a final illustration, we refer to some cases where one can compare the behavior of the \( M = 0 \) and \( M \neq 0 \) oscillators. During a large part of the period in the \( M = 0 \) oscillation the particle was closely and permanently sticking to the plate. One would expect that this is approximately true also for cases when \( M \) is not zero, but is rather small. Indeed, Figs. 6(d) and 8(d) depict this case, and the computation really predicts the particle and the plate moving close to each other. However, the enlarged details shown in Fig. 12(a) and (b) reveal that this approximately sticking motion consists of a number of successive impacts. This effect has been experienced by many experimentalists working with oscillating parts of some experimental devices as a side effect and actually as a nuisance. This is another illustration of the fact that chaos is lurking everywhere and that its manifestations await us in apparently the most simple physical situations.

Indeed, the chaotic behavior in the described system is not unexpected if one understands the particle and the plate as a system of two coupled oscillators. Each of them has its proper frequency, the mass \( M \) oscillating harmonically with the period \( T_{har} = 2\pi/\omega = 2\pi/\sqrt{M/k} \) and the particle oscillating with the gravitational period \( 2T_{gr} = 2\sqrt{2m/g} \). Since their ratio \( T_{har}/(2T_{gr}) = \pi \sqrt{(Mg)/(2\hbar\kappa)} = \pi \sqrt{\kappa/\mu} \) is irrational for almost all values of \( M \), we expect that this incommensurability leads to the chaotic dynamics for all calculated cases \( M \neq 0 \).

We note that investigations of chaotic properties of the nondissipative system described by the Hamiltonian (16) have not been previously reported. Searching through the literature, we found results on other systems which differ in some of their elements and only partially resemble our dynamical system. These investigations can be roughly systematized into three groups.

The first group follows from the original idea by Fermi\(^1\) modified so as to include gravitation.\(^2\) The resulting mechanism for accelerating particles is called the “gravitational engine” by Sagdeev, Usikov, and Zaslavsky.\(^26\) Into this group belong the papers by Lieberman and Lichtenberg,\(^4\) Tufillaro,\(^8\) Tufillaro and Albano,\(^9\) Mello and Tuffilaro,\(^10\) and Zimmerman and Celaschi.\(^11\) This work, however, concentrates on the small constant amplitude oscillations of an infinitely massive plate, which differs essentially from the energy conserving dynamics of the Hamiltonian (16).

The second group comprises the Bender bouncer of Ref. 27, the oscillator analyzed by Wai Chin et al.,\(^23\) and other similar systems.\(^28\) These are mostly one-body driven harmonic oscillators, which, at variance with our case, do not involve the gravitational force. To this category belongs also an interesting many-body system: the one-dimensional array of alternating free and harmonically bound particles. The free particles are bouncing elastically between the bound particles oscillating around their equilibrium positions. This is the so-called Visscher or ding-a-ling model, proposed by Ford\(^29\) and Casati et al.\(^30\) as a possible mechanism for explaining the diffusive energy transfer and the Fourier heat conductivity law.

The third group contains conservative systems with gravi-
Fig. 9. Several different maps reflecting the dynamics of the particle–plate bouncing problem and exhibiting typical attractors for $M/m = 100$, $M/m = 46.5$, and $M/m = 40$, with $\Omega = 6$. (a) The next return map for $\Delta t(n)$ where $\Delta t(n) = t(n) - t(n - 1)$; (b) the map connecting the amplitude of the plate oscillation with the time interval $A(n) \rightarrow \Delta t(n)$; (c) the next return map for $x(n)$; (d) the map connecting the impact position with the plate velocity immediately before the $n$th collision $v_y(n) \rightarrow x(n)$; (e) the next return map of the phase at the $n$th collision, with $f(n) = \phi_n / \pi$. 

tation but without harmonic force. It includes the many-body one-dimensional problem of Wojtkowski\textsuperscript{31} where $N$ balls with gravity move on a vertical line. The balls are colliding between themselves and with the hard horizontal floor. A similar problem is that of two balls with gravity explored by Whelan, Goodins, and Cannizzo.\textsuperscript{32} They examine the motion of two point particles on a vertical line, impacting with the hard floor and between themselves, in dependence on the mass ratio. We stress again, however, that in this type of motion there is no harmonic oscillation present.

That was a short and incomplete list of references on systems having some similarity to our particle–plate system. For many other aspects of chaotic behavior with gravitational and elastic forces involved, as well as for all other aspects of nonlinear dynamics, the reader is reminded of the extensive and useful list of references recently compiled by Hilborn and Tufillaro.\textsuperscript{33}

V. CONCLUSIONS

In this work we have investigated the conservative gravitationally driven stochastic Fermi oscillator, consisting of a particle of mass $m$ bouncing on a plate of mass $M$, supported by an elastic spring of constant $k$. The motion of the particle and the plate consists of a series of elastic impacts and, between them, of the smooth free fall motion and harmonic oscillations, respectively. The final result of these deterministic components is that the system as a whole behaves stochastically. The results obtained lead to following conclusions.

The behavior of the system is regular only in the case $M = 0$, when it reduces to a one-dimensional integrable problem. For $M \neq 0$ the motion is chaotic, the degree and character of chaotic dynamics essentially depending on the parameters $\mu = M/m$ and $k$ (or $\omega = \sqrt{k/M}$).

Generally, both phases and amplitudes differ from collision to collision. However, in certain regions of parameters the system has properties similar to those of the dissipative bouncing-ball problems with constant plate amplitude. An interesting attractor is revealed, changing as $\mu$ is varied from a simple one-dimensional curve to the complex object with fractal structure. At some critical value of $\mu$ the attractor is destroyed and fully chaotic behavior prevails. This suggests the existence of a complex bifurcation structure which is presently the subject of further investigations. In other parameter regions, the particle and the plate behave as a conservative system of two coupled oscillators. Position-velocity maps for certain frequency ratios clearly show the resonant...
islands of stability in the chaotic sea. The presence of both conservative and dissipative dynamics\(^1\) confirms the relevance of the problem considered here for further investigations.

The system considered is an example of chaotic behavior in the simple realistic system not far away from those considered in introductory physics courses. In the limit \(M = 0\) it offers a gradual transition from purely gravitational to purely elastic forces. Therefore we conclude with several suggestions concerning its inclusion in textbooks and teaching practice.

The case \(M = 0\) stresses the connection between the oscillatory motion and the shape of the potential energy curve. As an example it is appropriate for undergraduate introductory physics courses and for courses devoted to future physics teachers. In a simplified form it could also find its place in the secondary school curriculum. The phase portraits, illustrating the dependence of velocity on position in conservative systems, reflect in a similar way this connection. Their additional importance lies in the fact that, avoiding the explicit time dependence, they lead to path integrals as the appropriate way toward an early understanding of the ideas of quantization and the correspondence principle.

The case \(M \neq 0\) has a remarkable educational value because of its closeness to classical introductory physics subjects. It illustrates the energy transformations in many phenomena of everyday experience. Students are able to ask the relevant questions and to answer them themselves by discovering that the sequence of exactly deterministic solutions can end up in a chaotic series of data. Since the computing procedure does not involve differential equations but only the search for a root of a function, the numerical computation for many different sets of parameters and initial data can be reproduced on the undergraduate and sometimes even on the secondary school level. Finally, the results presented in this work show how the chaotic dynamics is reflected in different plots and maps, and can be used as a natural way for students to get acquainted with chaotic objects—attractors, fractals, and time series.

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