Sonic band structure and localized modes in a density-modulated system: Experiment and theory

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The sonic passing bands and stopping gaps of a quasi-one-dimensional air tube with modulated mass density were studied experimentally and theoretically. Some gap modes whose wave functions are strongly localized near the ends of the air tube were also found. The simple experiment can be used as a demonstration of band structure in an upper-division physics course.

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The Bloch theory indicates that the eigenfunctions of an electron in a periodic potential are extended and the corresponding eigenvalues form allowed energy bands and forbidden gaps. Recently, there has been much excitement in the physics community concerning the extension of the idea to electromagnetic and acoustic waves.[1] A photonic band material with periodically varying dielectric constant has been found,[2] and the vibrational properties of a finite one-dimensional string-mass chain has also been studied.[3] The vibrational passing bands and gaps were found experimentally and theoretically for a periodically loaded string. A randomly loaded string was also investigated. Some frequency bands and Anderson localized gap modes due to disorder were found. An interesting analogy between electron band structure and sonic band structure in fluids with periodic density variations was discussed theoretically by Dowling[4] using an infinite periodic model. A chain of delta functions was used to describe the mass density variation and Kronig–Penney-type solutions, similar to the Dirac comb problem in quantum mechanics, were obtained. A more comprehensive theoretical and experimental investigation on a similar one-dimensional system was carried out by Bradley.[5] Experimentally, an air-filled rectangle aluminum duct that was loaded with a periodic array of rectangle side branches was used to find the frequency band structure.

We discuss a *finite* quasi-one-dimensional system with arbitrary density modulation. This system allows us to extend the study of the eigenproperties to the inclusion of disorder in an acoustic system. Because the model is finite, it is possible to make direct comparisons with experiment. Also, because of the presence of the boundaries, the finite size effects on the band structure can be investigated.

In the experiment, a PVC tube with a wall thickness of 1.25 cm, length of 285.3 cm, and interior diameter of 14.3 cm was evenly divided by 16 baffles of 1.25 cm thickness. The baffles were made of particle board in the shape of a half-circle. The half-circle shape, instead of baffles with a central hole, was chosen because of a potentially interesting sound behavior due to the arrangement of baffles at an angle relative to one another. This arrangement of the baffles also makes the current system different from a "beads on a string" system.[3] The length of the acoustic chamber created by the baffles is 15.6 cm with a tolerance of 0.1 cm. The system is excited by a 5-in., 40-W loudspeaker fixed at one end of the tube. The speaker is controlled by a Tektronix FG 540 function generator amplified by an Altec 1040 B Solid State Amplifier. The function generator can deliver either short pulses or harmonic oscillations. Both normal mode analysis and pulse analysis were used. In the normal mode analysis, the transmitted sound intensity through the tube is recorded by a B&K 2209 sound meter while the harmonic frequency of the exciting source is being swept. In the pulse analysis, a pulse is delivered to the tube and the subsequent signal (a time series) is acquired with a Tektronix TDS 320 digital capture oscilloscope. The signal is then Fourier transformed and analyzed in the frequency domain using MATHCAD. The spectra obtained from both methods were compared with our theoretical predictions. In the experiment it was found that the results of the normal mode and pulse analysis were remarkably close, so in this paper only the normal mode results will be presented.

We use Dowling's simple theoretical model,[4] except that ours is finite. A d-function mass density is used to represent the baffles. This is a good approximation because the thickness of the baffles is much smaller than the length of the tube. In the model a velocity field is used to describe the pressure and velocity changes in a fluid system.[6,7] The one-dimensional velocity potential $\phi(x)$ satisfies the differential equation

$$\frac{d^2\phi(x)}{dx^2}+\omega^2\kappa\sigma(x)\phi(x)=0, \tag{1}$$

where

$$\sigma(x)=\sigma_0+\alpha\sum_{i=1}^{N}\delta(x-x_i). \tag{2}$$

In Eq. (2), σ_0 is the linear mass density of the air, α measures the mass density of the baffles, x_i indicates the locations of the baffles, N is the total number of baffles, and κ is the compressibility of the air. The velocity change along the x-direction is the negative gradient of the field $\phi(x)$ and is given by

$$v(x)=-\frac{d\phi(x)}{dx}. \tag{3}$$

For the present system, the two ends of the tube are rigid and stationary, and the field $\phi(x)$ must match the boundary condition[6]

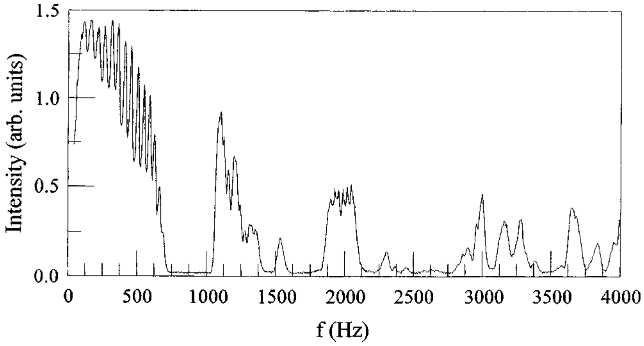

Fig. 1. The spectrum obtained by the normal mode analysis.

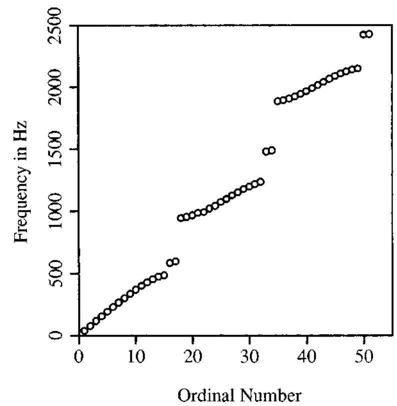

Fig. 2. Calculated eigenfrequencies for the tube with 16 evenly spaced baffles.

$$\frac{d\phi(x)}{dx}=0. \tag{4}$$

This boundary condition leads us to the following expansion of $\phi(x)$:

$$\phi(x)=\sum_{m=1}^{\infty}\sqrt{\frac{2}{L}}\cos\left(\frac{m\pi x}{L}\right)c_m, \tag{5}$$

where L is the total length of the tube. The boundary condition is satisfied automatically with the cosine wave expansion. In the numerical work, the upper limit of the expansion in Eq. (5) is $N_{\max}$, which is determined by the desired accuracy of the solution. It was found that 100 cosine waves ($N_{\max}=100$) lead to very high accuracy.[3]

If we substitute the expansion (5) into Eq. (1), we obtain the matrix equation

$$\sum_{m=1}^{N_{\max}}\left[-\left(\frac{m\pi}{L}\right)^2\delta_{m,n}+\sigma_{mn}\omega^2\right]c_m=0, \tag{6}$$

where

$$\sigma_{mn}=\kappa\int_0^L\sigma(x)\cos\left(\frac{m\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right). \tag{7}$$

The 100×100 matrix is diagonalized using the IMSL/Math/Library software package which yields eigenvalues and eigenfunctions as output.

For an empty air tube, the matrix equation results in a set of evenly spaced eigenvalues as expected. This trend in the frequency spectrum persists up to a critical frequency above which the cross waves, the waves along the radial direction, begin to set in. The radius of the cross section of the cylindrical tube is 7.2 cm, and the lowest cross-mode frequency is about 3000 Hz.[8]

To check our experimental methods, the tube was first excited without any baffles. The normal mode spectrum showed evenly spaced peaks and became complicated beyond 3000 Hz due to the mixing of the axial and cross modes. The peak separation is 60 Hz, which is in very good agreement with the theory. Figure 1 displays the results for the tube with 16 evenly spaced baffles. Because of the higher mass density of the baffles, we expect that all the evenly spaced resonant frequencies of the empty tube are suppressed except those modes whose nodes are located at the positions of the baffles. The lowest such mode is at about 1000 Hz, which corresponds to the 18th mode of the empty tube. Because of the suppression of the lowest 17 eigenfrequencies, a gap develops in which acoustic wave propagation is inhibited. In the spectra shown in Fig. 1, frequency gaps can be easily seen. The lowest passing band consists of 15 distinctive peaks. Two more such passing bands can also clearly be seen at higher frequencies. Our observations are similar to the electronic energy bands in solid state physics. However, there is a distinct difference between acoustic frequency bands and electronic energy bands: that is, the acoustic frequency bandwidth becomes narrower at higher frequencies and the opposite behavior is exhibited by the electronic energy bands. Because of the narrowing in bandwidth at higher frequencies, the 15 peaks in the higher bands are not distinguishable.

Figure 2 displays the calculated frequency versus the ordinal number of the eigenmodes of the baffled tube by solving the wave equation, Eq. (6). Three passing bands and stopping gaps are shown. The wave functions corresponding to the 15 eigenmodes in the passing bands are extended over the entire tube as expected. The frequencies are in good agreement with the observed data shown in Fig. 1. An interesting feature of Fig. 2 is the gap modes near 600, 1560, and 2500 Hz. From the calculation, the wave functions of the three modes are localized near the ends of the tube and they are nearly twofold degenerate. The characteristics of the localized modes are very similar to that of defect modes in a perfect lattice. We believe that the localized gap modes originate from the finite size of the tube: the ending points of the finite tube serve as "defects" in an otherwise perfectly periodic system.[9] Experimentally, two gap modes are shown by well-defined peaks near 1550 and 2200 Hz in the spectrum. However, the gap modes near 600 Hz are missing from the experimental spectrum, because they are too close to the upper edge of the lowest passing band.

In conclusion, an effective experimental method for studying the acoustic properties of a mass-density-modulated system was developed. In the system with periodic mass variation, some well-defined passing frequency bands separated by forbidden gaps were found. Also a number of interesting localized gap modes were investigated, which are reminiscent of defect modes in solid state physics.

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NATURE IS ALWAYS SMARTER

Perhaps the reason why science works, in the absence of a fixed method or a fixed set of rules, is that it is based on an ethic which recognizes that while any individual is obligated to champion what they honestly believe, no individual is the arbitrator of the correctness, or even the interest or usefulness of their own ideas. Experience teaches us that no matter how sure of ourselves we may feel, and how clever we may think we are being at certain instants, nature is always smarter, and anyone’s individual achievement may only survive to the extent to which it is superseded by the achievements of others.

Lee Smolin, *The Life of the Cosmos* (Oxford University Press, New York, NY, 1997), p. 297.