Common Substitutions and Approximations

Trig substitutions:

$$\sin^2 x + \cos^2 x = 1$$
$$e^{ix} = \cos x + i \sin x$$
$$\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) = \cos\left(\frac{\pi}{2} - x\right)$$
$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right) = \sin\left(\frac{\pi}{2} - x\right)$$
$$\sin(x+y) = \sin x \cos y + \cos x \sin y \Longrightarrow \sin 2x = 2 \sin x \cos x$$
$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$
$$\cos(x+y) = \cos x \cos y - \sin x \sin y \Longrightarrow \cos 2x = \cos^2 x - \sin^2 x$$
$$\cos(x-y) = \cos x \cos y + \sin x \sin y$$

Approximations:

All of the following series approximations can be derived from Taylor's series:

$$f(x+\epsilon) = f(x) + \epsilon f'(x) + \frac{\epsilon^2}{2!} f''(x) + \frac{\epsilon^3}{3!} f'''(x) \cdots$$

So for $|\epsilon| \ll |x|$,

$$f(x+\epsilon) \approx f(x) + \epsilon \frac{d}{dx} f(x) \ .$$

Note: all angles must be in radians!

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$
$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \cdots \quad \left(x^2 < \frac{\pi^2}{4}\right)$$
$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$
$$(x+y)^n = x^n + nx^{n-1}y + \frac{n(n-1)}{2!}x^{n-2}y^2 + \frac{n(n-1)(n-2)}{3!}x^{n-3}y^3 \cdots \quad (y^2 < x^2)$$
$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \cdots$$
$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 \cdots \quad (|x| < 1)$$

So for $|x| \ll 1$

$$\sin x \approx x$$
$$\tan x \approx x$$
$$\cos x \approx 1$$
$$(1 \pm x)^n \approx 1 \pm nx$$
$$e^x \approx 1 + x$$
$$\ln(1 + x) \approx x$$

Stirling's approximation:

$$\ln(K!) \approx K(\ln K - 1)$$