cated harmonic oscillator potential of the same range provides a convenient and instructive example of a quantum mechanical system for which exact bound-state and positive-energy solutions may be obtained in closed form. In a pedagogical study of tunneling, the model provides a graphical example of the way in which the positive-energy components of a wave packet "leak" through the barrier, leaving the bound-state component behind. Finally, an interference between the bound and continuum states is shown to underlie oscillatory variations in the probability of the particle being found inside the potential barrier.

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The effects of coefficient of restitution variations on long fly balls

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The coefficient of restitution of major league baseballs is required to be 0.546 ± 0.032. These allowed variations affect the launch velocity and ultimately the range of fly balls. The variations in the range for well-hit balls are calculated here to be on the order of 15 ft. These calculations provide an interesting collection of mechanics problems that might be of interest for undergraduate students.

I. INTRODUCTION

The coefficient of restitution for a collision is the ratio of the final relative velocity to the initial relative velocity of the colliding objects. The coefficient of restitution of a major league baseball is required to be 0.546 ± 0.032.[1,2] The question raised here is how will the allowed variations in the coefficient of restitution affect the distance traveled by a well-hit ball?

The first step toward the answer is to find the effect of coefficient of restitution variations on the velocity of the ball as it leaves the bat, referred to here as the launch velocity. Next, the range for a baseball as a function of the launch velocity must be studied. To keep some resemblance to reality, air resistance must be included. Finally, the two results can be put together to find the variations in the range due to fluctuations in the coefficient of restitution. Since the interest here is just the variations in the range, as opposed to the values of the range, and because it is often easier and frequently more accurate to calculate variations, as opposed to the actual values themselves, only the variations will be found.

II. THE VARIATION OF THE LAUNCH VELOCITY WITH THE COEFFICIENT OF RESTITUTION

The collision viewed from a coordinate system moving with the center of mass of the bat at the moment just before impact is shown in Fig. 1. According to the standard assumptions, the bat can be treated as a free body.[3] The equations needed to describe the ball–bat collision come from the conservation of linear momentum, the conservation of angular momentum, and the definition of the coefficient of restitution. These equations are

$$mv_0 = MV - mv, \tag{1}$$

$$I\omega_0 - mv_0 b = I\omega + mvb, \tag{2}$$

$$e = (v + V - \omega b)/(v_0 + \omega b), \tag{3}$$

where m is the mass of the ball, M is the mass of the bat, I is the moment of inertia of the bat about the center of mass, v_0 and v are the initial and final velocities of the ball, V is the final velocity of the center of mass of the bat, ω_0 and ω are the initial and final angular velocities of the bat, b is the

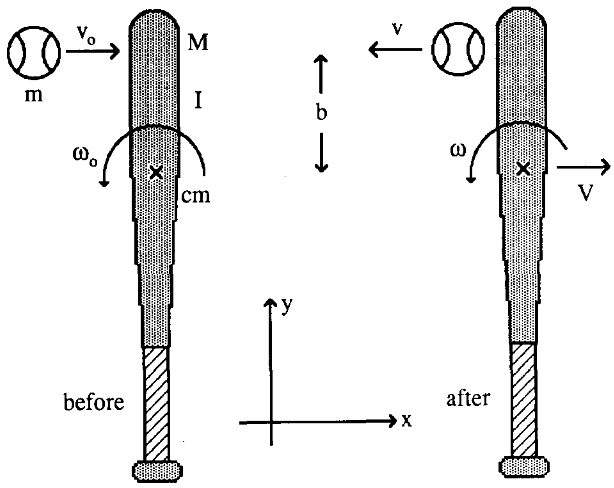

Fig. 1. The ball–bat collision viewed from a coordinate system moving with the center of mass of the bat at the moment just before impact.

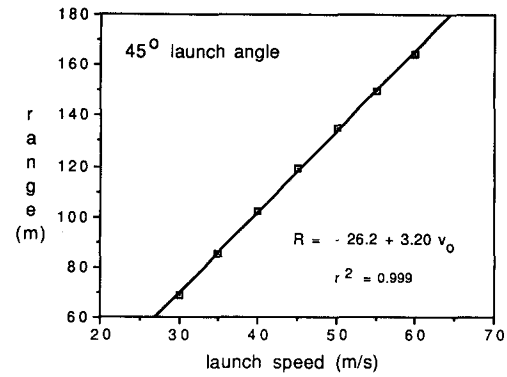

Fig. 2. Range versus launch speed for 45° including the least-squares-fit line.

distance between the center of mass of the bat and the point where the ball collides, and, finally, e is the coefficient of restitution. Equations (1), (2), and (3) assume that all the motion is happening in a plane, which is only approximately correct.

Eliminating V and ω and solving for the final velocity of the ball gives

$$v = [I\omega_0 b(1+e) - mb^2 v_0 - I v_0 (m/M - e)] \times [mb^2 + I(1+m/M)]^{-1}. \quad (4)$$

This is the launch velocity of the ball in the c.m. frame. Equation (4) can be differentiated to find the variation of the launch velocity of the ball with the coefficient of restitution,

$$\frac{dv}{de} = \frac{\omega_0 b + v_0}{mb^2/I + (1+m/M)}. \quad (5)$$

Using some typical values such as $m \approx 0.15$ kg, $M \approx 1.0$ kg, $b < 0.10$ m, $v_0 \approx 50$ m/s, $\omega_0 \approx 10$ rad/s, and $I \approx 0.05$ kg m^2, the terms in Eq. (5) are found to be on the order of

$$\omega_0 b \approx 1 \text{ m/s},$$
$$v_0 \approx 50 \text{ m/s},$$
$$mb^2/I \approx 0.03,$$
$$1 + m/M \approx 1.2.$$

Table I. The range in meters for a baseball calculated for various angles and launch speeds in meters per second. The calculations were done using $\mathbf{a} = -\mathbf{g} - cv\mathbf{v}$, where $c = 0.0050\ \text{m}^{-1}$.

v	30°	35°	40°	45°	50°	55°	60°
30	63.248	67.018	69.327	68.712	66.800	63.535	57.895
35	79.982	83.443	86.893	85.379	83.716	79.050	72.788
40	97.342	102.185	103.210	102.367	99.582	94.779	86.862
45	112.931	119.295	121.277	119.388	115.409	109.254	99.814
50	130.664	134.463	137.533	134.708	131.006	123.475	112.517
55	146.189	151.331	151.832	149.722	144.918	136.167	124.870
60	161.502	166.018	167.476	164.353	158.428	148.463	135.827

The point is that the first term in the numerator and the first term in the denominator of Eq. (5) can both be neglected, leaving the result

$$\frac{dv}{de} = \frac{v_0}{1 + m/M}. \quad (6)$$

It is interesting to note that this is the same result as for the collision of two billiard balls. This is due to the fact that the collisions where the ball is well hit occur near the center of mass of the bat.

It is easiest to transform out of the c.m. frame and into the "home plate" frame at this point. The variation of the launch velocity with the coefficient of restitution is now given by

$$\frac{dv}{de} = \frac{v_0 + v_b}{1 + m/M}, \quad (7)$$

where v is now the launch velocity, v_b is the speed of the center of mass of the bat, and v_0 is now the speed of the pitch. It will be assumed that Eq. (7) is valid even when the ball is launched at an angle.

III. THE RANGE AS A FUNCTION OF THE LAUNCH VELOCITY

The range of a projectile in the absence of air resistance is given by

$$R = (v^2 \sin 2\theta)/g.$$

However, air resistance cannot be neglected in the flight of baseballs. Assuming that the air resistance is proportional to the velocity squared, the components of the acceleration of the ball can be written as

$$a_x = -cvv_x,$$
$$a_y = -g - cvv_y,$$

where g is the acceleration due to gravity, v is the speed of the ball, v_x and v_y are the horizontal and vertical components of the velocity of the ball, and c is the drag factor.

Using the value $c = 0.0050$/m given by Brancazio,[4] the range of the ball for various launch speeds and firing angles was calculated numerically (e.g., see Swartz).[5] The results are summarized in Table I. Notice that the maximum range does not occur at the 45° from the no-air resistance case, but is closer to 40°. This is consistent with the idea that the air resistance causes a projectile to "run out of gas" and "drop like a rock." That is why outfiedlers are taught to get "under the ball."

Figure 2 is a typical plot of the range versus the launch speed for a given launch angle. The slope of the curve generated by these points is the variation of the range with the launch speed. The calculated points are remarkably close to a straight line. While least-squares fitting is usually reserved for fitting data, as opposed to calculated points, it provides a convenient method for finding the slope. Note that the points do not form a straight line exactly. They are above the line in the middle and below at the extremes.

Table II contains the least-squares fit values of the slopes as well as the correlation coefficients for the fits. The slopes vary from 2.60 to 3.32 s with the angle while the correlation coefficients are surprisingly high considering that the range of launch speeds covers a factor of 2.

IV. THE VARIATIONS IN RANGE DUE TO FLUCTUATIONS IN THE COEFFICIENT OF RESTITUTION

The variations in the range due to the fluctuations in the coefficient of restitution can now be found in a straightforward manner,

$$\frac{dR}{de} = \left(\frac{dR}{dv}\right)\cdot\left(\frac{dv}{de}\right). \tag{8}$$

While the calculation indicated by Eq. (8) can be repeated for many combinations of pitch speed, bat speed, and launch angle, it is more instructive just to use some typical values to get an idea of the size of the effect. Using an average value of 3 s for the variation of the range with the launch speed and the variation of the launch speed due to the coefficient of restitution from Eq. (7) yields

$$\frac{dR}{de} \approx (3\ \text{s})\cdot\frac{v_0 + v_b}{1 + m/M}. \tag{9}$$

Typical values of $v_0 + v_b \sim 60$ m/s, $m \approx 0.15$ kg, and $M \approx 1.0$ kg give a value of

$$\frac{dR}{de} \approx 160\ \text{m}.$$

The value allowed by major league baseball of $de = 0.032$ results in range variations of

$$dR \approx 5.1\ \text{m} \approx 17\ \text{ft!}$$

This is about equal to the width of the warning track.

Table II. The least-squares-fit slopes for the range versus launch speed plots at various angles.

Launch angle	Slope $= \frac{dR}{dv}$	Correlation coefficient r^2
30°	3.29 s	1.000
35°	3.32 s	0.999
40°	3.28 s	0.999
45°	3.20 s	0.999
50°	3.06 s	0.999
55°	2.84 s	0.998
60°	2.60 s	0.998

V. CONCLUSIONS

These results bring up some interesting (if meaningless) questions. Is the difference between a warning track out and a home run really influenced by slight variations in the baseball? Is there a way that a pitcher can tell before he throws a pitch if he has a live or a dead ball? Could Dennis Eckersley have thrown Kirk Gibson, in the vernacular of baseball, "a good ball to hit"?

In all seriousness, the physics problem described here is a good fraction of an upper division undergraduate mechanics course. Its solution involves rotational collisions, the application of the conservation laws for angular and linear momentum, the concept of the coefficient of restitution, transformations to and from center-of-mass coordinates, projectile motion including air resistance, numerical analysis of motion, least-squares fitting, and, perhaps most importantly, it produces interesting results!

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APPENDIX

It is appropriate to make some comments on the limitations of the model presented here. The value of the coefficient of restitution specified by the major leagues is prescribed to be measured by firing a ball at 85 ft/s (≈ 56 mph ≈ 26 m/s) at a wall of ash, the material of which the bats are constructed. The value of the coefficient of restitution is then the ratio of the rebound to incident velocities.[2] This value may be different from the value in a ball–bat collision due to several factors. First, the bat may absorb energy differently than the wall of ash (for example, by vibrating). Second, the actual relative velocities during the collisions discussed here are much higher than 26 m/s. The coefficient of restitution of a baseball is known to vary with impact velocities.[6] In general, the higher the impact velocity the lower the coefficient of restitution. Third, the tolerance in the coefficient of restitution of ± 0.032 is not necessarily a manufacturing tolerance. That is, the variations from ball to ball may actually be much smaller than this.

Approximations have also been made in describing the flight of the ball. The drag factor for a baseball is not well known. In addition, and perhaps more importantly, it has been suggested that the drag factor varies dramatically during the flight as the air passing the ball changes from partially turbulent flow to fully turbulent flow and back.[7,8] Also, the spin imparted to the ball during the collision has a substantial effect on the range.[9]

In summary, the answers to the equations raised in the conclusion are "probably not," "almost certainly not," and "definitely not, the baby cleared the fence by 25 ft!"

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A short proof of the generalized Helmholtz theorem

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By making use of the Stokes operator and its corresponding dual operator, which are introduced here, a short proof of the Helmholtz theorem for antisymmetric second-rank tensor fields in Minkowski space-time is presented.

One of the most straightforward theoretical presentations of Maxwell's equations is based on the Helmholtz theorem for antisymmetric second-rank tensor fields in Minkowski space-time (M^4).[1–3] Kobe[2] has given the following formulation for this generalized Helmholtz theorem: An antisymmetric second-rank tensor field in M^4 that vanishes at spatial infinity is completely determined by specifying its divergence and the divergence of its dual. In particular, if $F^{\mu\nu}(x)$ is an antisymmetric second-rank tensor field[4] that vanishes at spatial infinity and if its divergence $\partial_\mu F^{\mu\nu}$ is specified to be the charge-current density four-vector (times $4\pi/c$), and the divergence of its dual $\partial^*_\mu F^{\mu\nu}$ vanishes everywhere in M^4, then $F^{\mu\nu}(x)$ is completely determined and it is the electromagnetic field tensor, i.e. Maxwell's equations are obtained. The purpose of this article is to present an alternative proof of the generalized Helmholtz theorem[5] that is shorter than the proof given by Kobe.[2]

A useful operator for stating the generalized Helmholtz theorem is introduced here. It is the Stokes operator or Stokian $\partial^{\mu\nu}_\lambda$, which is antisymmetric in μ and ν, and is defined as

$$\partial^{\mu\nu}_\lambda = \delta^\nu_\lambda\,\partial^\mu - \delta^\mu_\lambda\partial^\nu. \tag{1}$$

The operator $*\partial^{\mu\nu}_\lambda$, antisymmetric in μ and ν, dual to $\partial^{\mu\nu}_\lambda$ is defined as

$$*\partial^{\mu\nu}_\lambda = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}\,\partial_{\alpha\beta\lambda}, \tag{2}$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the completely antisymmetric Levi–Civita tensor with $\epsilon^{0123}=1$. The operator $*\partial^{\mu\nu}_\lambda$ has the following properties:

$$*[*\partial^{\mu\nu}_\lambda] = -\partial^{\mu\nu}_\lambda \tag{3}$$

and

$$\partial_\mu[*\partial^{\mu\nu}_\lambda] = 0. \tag{4}$$

Now, the generalized Helmholtz theorem may be stated as follows: An antisymmetric second-rank C^1-tensor field[6] $F^{\mu\nu}(x)$ in M^4 that vanishes at spatial infinity can be expressed as[2]

$$F^{\mu\nu}(x) = \partial^{\mu\nu}_\lambda A^\lambda - *\partial^{\mu\nu}_\lambda B^\lambda, \tag{5}$$

where $A^\lambda(x)$ is a four-vector of class C^2 in M^4 that is defined as

$$A^\lambda = \partial_\alpha Z^{\alpha\lambda}, \tag{6}$$

and $B^\lambda(x)$ is another independent four-vector of class C^2 in M^4 that is defined as

$$B^\lambda = \partial^*_\alpha Z^{\alpha\lambda}. \tag{7}$$

The tensor $Z^{\alpha\lambda}(x)$ in Eq. (6) is an arbitrary antisymmetric second-rank C^2-tensor field in M^4 that vanishes at spatial infinity, and the tensor $*Z^{\alpha\lambda}(x)$ in Eq. (7) is its corresponding dual, which is defined as

$$*Z^{\mu\nu}(x) = \tfrac{1}{2}\epsilon^{\mu\nu\alpha\beta}Z_{\alpha\beta}(x). \tag{8}$$

The proof of Eq. (5) follows directly from the tensor identity

$$\partial_\alpha\,\partial^\alpha Z^{\mu\nu} = \partial^{\mu\nu}_\lambda[\partial_\alpha Z^{\alpha\lambda}] - *\partial^{\mu\nu}_\lambda[\partial^*_\alpha Z^{\alpha\lambda}]. \tag{9}$$

This identity is proved in the Appendix. In fact, if the tensor field $F^{\mu\nu}(x)$ is taken to be

$$F^{\mu\nu}(x) = \partial_\alpha\,\partial^\alpha Z^{\mu\nu}, \tag{10}$$

then the substitution of Eqs. (6), (7), and (10) into Eq. (9) leads to the expression (5) for the tensor field $F^{\mu\nu}(x)$. Thus the generalized Helmholtz theorem shows that the tensor field $F^{\mu\nu}(x)$ can be resolved into the sum of the Stokian of a four-vector and the dual of the Stokian of another independent four-vector.

A corollary of the generalized Helmholtz theorem states that the tensor field $F^{\mu\nu}(x)$ is determined by specifying its divergence $\partial_\mu F^{\mu\nu}$ and the divergence of its dual $\partial^*_\mu F^{\mu\nu}$ everywhere in M^4. The proof of this corollary is straightforward. The divergence of Eq. (5) leads to the equation

$$\partial_\mu\,\partial^\mu A^\nu - \partial^\nu\,\partial_\mu A^\mu = \partial_\mu F^{\mu\nu}. \tag{11}$$

This result has been obtained by making use of the identity (4). Now, from Eq. (6), it follows that $\partial_\mu A^\mu = 0$ and hence Eq. (11) reduces to the d'Alembert equation

$$\partial_\mu\,\partial^\mu A^\nu = \partial_\mu F^{\mu\nu}. \tag{12}$$

Specifying $\partial_\mu F^{\mu\nu}$ in M^4 and imposing the condition that A^ν

