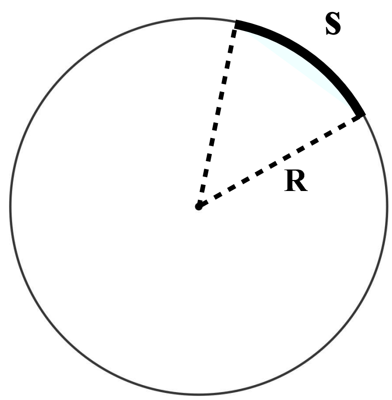
CSUC
Department of Physics
301A Mechanics

I. SOLID ANGLE AND AVERAGING OF HARMONIC FUNCTIONS

A. Solid angle: A simple generalization of the angle concept

1. Planar angle

In its essence, the concept of *angle* is the notion of measuring the portion of a circle one has encompassed in traversing a distance s around its circumference.



The most natural *number* to associate with the portion of a circle we have swept out on marching a distance s about its circumference would seem to be the "Greek" (philosophically simple ...) choice:

$$\frac{s}{circumference}$$

This is commonly used today in the concept e.g. of "frequency", as in "the number of revolutions per time". With this designation of the measure of *angle*, one complete circle is then represented by unity.

More common in scientific discussions is to use the number:

$$2\pi\left(\frac{s}{circumference}\right) = \frac{s}{R}$$

which we call the *radian measure* associated with the angle and is actually the physical distance traversed along the arc if our circle is a unit circle. This means we would now represent one complete circle by the number 2π . A third common choice is the Babylonian 'astrological' choice or "degree measure" which arbitrarily divides the circle into 360 parts and which the ancients chose because 360 is roughly the number of days in a year. This number is then:

$$360\left(\frac{s}{circumference}\right)$$

2. Solid angle

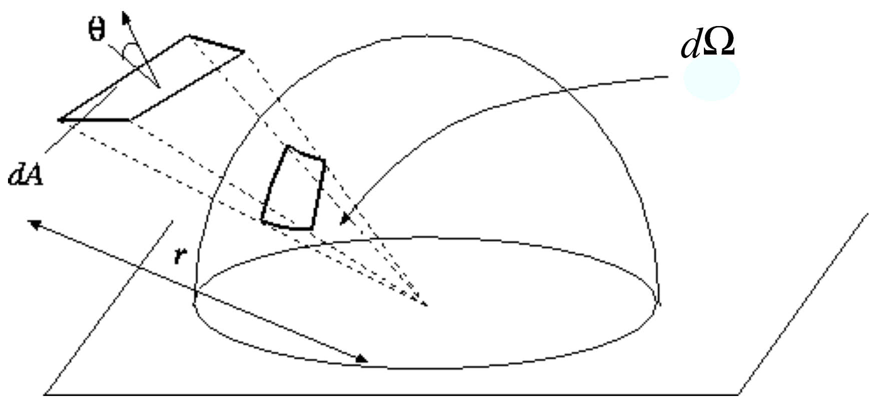
The concept of solid angle lies in the motion of measuring the portion of a sphere one has encompassed with a given *area* A on the sphere's surface. The most natural measure would, again, surely seem to be ...

$$(Given\ Area)/(Total\ Surface\ Area).$$

More common, however, is to use the number which is the *exact analog* of radian measure, i.e.

$$4\pi \left(\frac{Given\ Area}{Total\ Surface\ Area} \right) = \frac{4\pi A}{4\pi R^2} = \frac{A}{R^2} \tag{1}$$

where the equation (1) is defined as the *steradian* measure. The universal symbol for solid angle is the Greek capital Ω. Notice, if you have a *unit* sphere then any specified surface area on it is numerically equal to the solid angle "subtended". One useful (but surely not unique) visualization of solid angle is the size of the 'cone' needed to encompass the given area.



Observe,as well, that solid angle (like planar angle) is dimensionless. If we were to stand at the sphere's very center, then a solid angle measures the portion of the total 'sky' intercepted or "blocked out". Notice especially, in spherical coordinates that the volume element

$$dV = r^2 dr \sin(\theta)\ d\theta\ d\phi$$

can be seen to be $dV = r^2 dr \cdot$ (infinitesimal area on a unit sphere), so in these coordinates, indeed,

$$dV = r^2 dr\ d\Omega$$

where

$$d\Omega = \sin(\theta)\ d\theta\ d\phi$$

The great utility of the notion of 'solid angle' is that it helps us to talk about the concept of *direction* in a coordinate free way — and to write down physical integrals over direction. Imagine dividing the surface of a unit sphere into very many infinitesimal patches. Then, if we take our origin at the center, all possible directions can be identified by selecting out some specific patch (i.e. we have discretized "direction"). We say, in fact, that there are ' 4π possible directions,' meaning that the sum total of the patches is 4π. Each patch identifies an infinitesimal cone of directions — this is ideally suited to integral calculus, which after all, sums up little portions of things. One often sees the notation $d\Omega_{\hat{n}}$ or $d\Omega\,(\hat{n})$ signifying a small cone of directions centered about the direction $\hat{n}$. For example, if we are faced with a surface integral over a sphere, it is helpful to recognize that $d\sigma = R^2 d\Omega$ where, on a sphere the radius R is constant: e.g.

$$\oint d\,(Area) = \int_{all\ dir} R^2 d\Omega = R^2 \int d\Omega = R^2 4\pi$$

B. Averaging property for solutions of Laplace's equations

Preliminaries : Suppose we let $\vec{r} = \vec{r_o} + R\hat{n}$ where $\vec{r_o}$ is a fixed vector. Then, for a given value of R, if we allow $\hat{n}$ to point in all possible directions, we observe that $\vec{r}$ traces out the surface of a sphere of radius R and center $\vec{r}_o$. For a given function ϕ we have $\phi(\vec{r}) = \phi(\vec{r}_o + R\hat{n})$, then by the multi-variable chain rule we observe that

$$\frac{d}{dR}\phi(\vec{r}_o + R\,\hat{n}) = \hat{n} \cdot \vec{\nabla}\phi(\vec{r})$$

Next, consider a spherical volume of space V of radius R centered on $\vec{r}_o$, and suppose that a function ϕ has the property that $\nabla^2\ \phi\ =\ 0$ throughout this volume, i.e. it obeys the Laplace equation. Then, its integral is certainly also zero ... i.e.

$$0 = \int_V dV \nabla^2 \phi = \int_V dV \vec{\nabla} \cdot \vec{\nabla}\phi = \oint_\Gamma \vec{\nabla}\phi \cdot \hat{n} d\sigma$$

We denote the surface of the volume by the symbol Γ and we have transformed the volume integral into a surface integral using the divergence theorem.. Suppose next we examine the special case where this surface Γ is the boundary of a *sphere* of radius R centered at $\vec{r}_o$. On the sphere's surface each position is given by

$$\vec{r} = \vec{r}_o + R\,\hat{n}$$

with $\hat{n}$ pointing in turn to each location on the surface. But since, as we just showed ...

$$\hat{n} \cdot \vec{\nabla}\phi(\vec{r}) = \frac{d}{dR}\phi(\vec{r}_o + R\,\hat{n})$$

then

$$0 = \oint_\Gamma d\sigma \frac{d}{dR}\phi(\vec{r}_o + R\,\hat{n}) = \int_{dir} R^2 d\Omega_{\hat{n}} \frac{d}{dR}\phi(\vec{r}_o + R\,\hat{n})$$

where we have used $d\sigma = R^2 d\Omega$ as long as we are on a sphere. Since R isn't a variable quantity in the integral we may pull the derivative outside the integration, finally yielding:

$$0 = R^2 \frac{d}{dR} \int_{dir} d\Omega_{\hat{n}} \phi(\vec{r}_o + R\,\hat{n}) \implies \int_{dir} d\Omega_{\hat{n}} \phi(\vec{r}_o + R\,\hat{n}) = const.$$

We are allowed to conclude that this integral is a constant which does *not* depend on R ... and so we *may as well* set $R = 0$!

$$\int d\Omega_{\hat{n}} \phi(\vec{r}_o + R\hat{n}) = \int d\Omega_{\hat{n}} \phi(\vec{r}_o) = 4\pi\phi(\vec{r}_o)$$

which implies, in summary then ...

$$\oint_\Gamma d\sigma \phi(\vec{r}) = R^2 \int d\Omega_{\hat{n}} \phi(\vec{r}_o + R\hat{n}) = 4\pi R^2 \phi(\vec{r}_o)$$

which we see implies:

$$\frac{1}{4\pi R^2} \oint d\sigma \phi(\vec{r}) = \phi(\vec{r}_o)$$

The conclusion is, then, if a function obeys the Laplace equation in a region of space it must also obey this amazing averaging property. Average the function over a spherical surface and you end up with the value of the function at its center!