Uncertainty in Measurement


INTRODUCTION

Since the analysis of uncertainties is essential to any scientific measurement, many of the laboratory exercises in 204A emphasize the estimation and propagation of experimental uncertainty. In this lab, we investigate what is meant by experimental uncertainty, we develop techniques for estimating uncertainty and for using uncertainties in calculations.

WHAT IS UNCERTAINTY?

If you were to measure the width of this piece of paper using a ruler, you would likely find it to be around 21.6 cm. But, is the paper exactly 21.6 cm? A careful ruler-user can probably consistently distinguish 21.6 cm from 21.7 cm, but can unlikely tell a 21.60 cm sheet from 21.61 cm. It would be foolish, using the same ruler, to say that the sheet of paper is 21.6030524 cm wide. The best we can say (with a ruler) is that the the width of the paper is closer to the 21.6 cm mark than any other mark. It's our best guess for the width. Taking the resolution of the ruler to be 0.1 cm, we can say explicitly that the width is between 21.55 cm and 21.65 cm.

When stating measured values, we state our best guess for the measurement $x_{\text{best}}$ and its uncertainty $\delta x$ as

$$x_{\text{best}} \pm \delta x$$

The quantity $\delta x$ is called the uncertainty of the measurement.

For the width $w$ of this paper, the measurement is correctly written as

$$w = 21.6 \pm 0.05 \text{ cm}$$

If a careful experimenter wanted to make a better measurement, she could use a better instrument, and get a result with a smaller uncertainty. But does she know how wide the sheet of paper is exactly? That's a tough question. The width is likely to be different at the top of the sheet than at the bottom, or near the middle. With very precise measurements, mechanical vibrations cause trouble. Variations in humidity and temperature cause the sheet to grow and shrink. Microscopically, thermal energy keeps the atoms in the paper constantly vibrating and moving. It's impossible to define exactly what we mean by the width of the paper— it's not a well-defined quantity. So, we say what we know: our best guess and its uncertainty: $w = 21.6 \pm 0.05$ cm. Uncertainties are usually rounded to one significant figure.

I should briefly mention systematic errors, which are generally caused by bad measurement equipment. Anytime an instrument has an out-of-date calibration, or is otherwise unverified, the experimenter runs the risk of taking inaccurate measurements. In the undergraduate lab, systematic errors
are not typically encountered. But in the real world, systematic errors happen and must carefully be avoided in experimental measurements.

**KEY IDEAS**

1. No measurement can be made with absolute certainty.

**IT’S ALL ABOUT COMPARISONS**

If I say that it’s 311 Kelvin outside today, that my dog has a mass of 1.3 slugs and that I ate a 2.5 megajoule breakfast today, it probably doesn’t mean much to you. These unfamiliar units illustrate an important point about measurements: they allow us to make comparisons. Using familiar units, I’d say it is 100°F today, my dog weighs 41 lbs, and I ate a 600 Calorie breakfast. This allows you to make a comparison to warmer or colder days, bigger or smaller dogs and to conclude that I should eat a healthier breakfast. Comparisons are the reason that we take any measurement. Any single measured quantity is completely uninteresting.

In lab, we compare our measurements to accepted values or to a value predicted by a physical model. Two measurements correctly stated with uncertainty, are *consistent* with one another when they are equal within their uncertainties. For two values $p$ and $q$, does

$$p_{\text{best}} \pm \delta p = q_{\text{best}} \pm \delta q ?$$

If this equality is true, $p$ and $q$ are said to be consistent.

As experimenters, we strive to make measurements of high precision. We want to be *certain* that measurements are consistent with predictions. So, measurements of low uncertainty are preferred. But avoid the temptation to underestimate uncertainty, it’s better to be unsure than wrong.

**KEY IDEAS**

1. Any single measured quantity is completely uninteresting.

2. We use a measurement and its uncertainty to *make a comparison* between the measurement and the accepted value or prediction.

3. It is better to be accurate than precise.

**TODAY’S GAME**

**GAME RULES**

In today’s lab, we will be playing a game. The goal of the game is to make *accurate* measurements with the *lowest possible uncertainty* for the instrument and technique used. You will be instructed to make several measurements using different techniques. With your lab group, make your best measurement and estimate its uncertainty. Higher precision answers are rewarded with more points, while inaccurate answers receive no points. You will then bring your measurement to the lab instructor, where it will be compared against a very high precision measurement. Consistent measurements will be awarded using the formula

$$\text{awarded points} = \sqrt{\frac{x_{\text{best}}}{\delta x}} - 1$$

**Equation 1.**

Measurements inconsistent with the high-precision standard measurement will receive 0 game points.
Payouts for select uncertainties

<table>
<thead>
<tr>
<th>% Uncertainty ($x_{\text{best}}/\delta x \times 100$)</th>
<th>$\sqrt{x_{\text{best}}/\delta x} - 1$</th>
<th>Game Points</th>
</tr>
</thead>
<tbody>
<tr>
<td>25%</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>10%</td>
<td>2</td>
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</tr>
<tr>
<td>5%</td>
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<td>9</td>
<td>9</td>
</tr>
<tr>
<td>0.1%</td>
<td>30</td>
<td>30</td>
</tr>
<tr>
<td>Measurement not in agreement</td>
<td>0</td>
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</tbody>
</table>

Explicitly follow the instructions for each measurement technique. Do not use measurement tools until instructed to do so.

★ The lab group with the most points at the end of the lab-period will win a prize ★

GAME ROUND 1: ESTIMATING INSTRUMENT UNCERTAINTY

METAL BLOCK

The uncertainty of a measurement depends on the tool used and the skill of the experimenter. Work as a group to estimate the uncertainty in the length offered by three measurement techniques.

SUBMISSION B1: State the length (with uncertainty) of the longest side of the block with no external tools or objects used for scale. You may handle the block if you like. Submit your measurement on a post-it note. Inches or centimeters are acceptable units. Calculate the payout (using Equation 0.1) you will receive if your measurement and uncertainty are consistent with the accepted value. The submission should include the block letter, your estimate of the length, the uncertainty, and the payout amount.

SUBMISSION B2: Measure the length of the longest side of the block in centimeters using the ruler. Calculate the payout (using Equation 0.1) you will receive if your measurement and uncertainty is consistent with the accepted value. Submit your measurement and its uncertainty on a post-it note. The submission should include the block letter, your measurement of the length, the uncertainty, and the payout amount.

SUBMISSION B3: Measure the length of the longest side of the block using the vernier calipers. Calculate the payout (using Equation 0.1) you will receive if your measurement and uncertainty is consistent with the accepted value. Submit your measurement and its uncertainty on a post-it note. The submission should include the block letter, your measurement of the length, the uncertainty, and the payout amount.

UNKNOWN MASS

SUBMISSION M1: State the mass (with uncertainty) of the provided mass with no external tools or objects used for scale. You may handle the mass if you like. Submit your measurement on a post-it note. Grams, kilograms, ounces or pounds are acceptable units. The submission should include the block letter, your estimate of the mass, the uncertainty, and the payout amount.
letter, your estimate of the mass, the uncertainty, and the payout amount.

**Submission M2:** Measure the mass provided in grams by constructing a meter-stick balance, and using the standard weights provided. Submit your measurement and its uncertainty on a post-it note. The submission should include the mass letter, your measurement of the mass, the uncertainty, and the payout amount.

**Submission M3:** Measure the mass provided in grams using the triple beam balance. Submit your measurement and its uncertainty on a post-it note. The submission should include the mass letter, your measurement of the mass, the uncertainty, and the payout amount.

**Lens Focal Length**

Sometimes the uncertainty of a measurement is not limited by the scale of the instrument used. For example, when a lens forms an in-focus image of a distant object onto a screen, the distance between the screen and the lens is called the *focal length*. When we try to measure this focal length, it may be difficult to identify the center of the lens, or determine exact in-focus position for the screen.

**Submission L1:** Place a piece of white paper on the table. Position the lens above the paper, and adjust the paper-to-lens distance required to form an image (on the paper) of the fluorescent lights overhead. Using a ruler, measure the this paper-to-lens distance in centimeters. Submit your measurement and its uncertainty on a post-it note. The submission should include your measurement of the length, the uncertainty, and the payout amount.

**Key Ideas**

1. It is important to make a reasonable estimate of the uncertainty of the measurement technique used.

2. A measurement is often more uncertain than the scale on an instrument suggests

**Using Uncertainties in Calculations**

The following formulas are reproduced, without proof, from Taylor (2007) cited above. These formulas are shown for three variables $x, y, z$ but can easily be generalized for more (or fewer) variables. If the various quantities $x, y, z$ are measured with small uncertainties $\delta x, \delta y, \delta z$, and the measured values are used to calculate some quantity $q$, then the uncertainties in $x, y, z$ cause an uncertainty in $q$ as follows:

**Error Propagation Formulas**

If $q$ is a sum or difference like

$$q = x + y - z,$$

$$\delta q^* = \sqrt{(\delta x)^2 + (\delta y)^2 + (\delta z)^2}$$  \hspace{1cm} (0.2)

*For non-random or non-independent uncertainties, we note that $\delta q \leq \delta x + \delta y + \delta z$ always. If you want to be cautious, you can just add the uncertainties directly.*
Example 1

The total mass of a bag of apples is \( m_t = 2.1 \pm 0.1 \) kg. The grocer tells you that the mass of the bag is \( m_b = 0.22 \pm 0.03 \) kg. What is the mass (and uncertainty) of the apples alone (\( m_a \))?

\[
m_a = m_t - m_b = 2.1 \text{ kg} - 0.22 \text{ kg} = 1.88 \text{ kg}
\]

From Equation 0.2 we calculate the uncertainty in the apple mass to be

\[
\delta m_a = \sqrt{(\delta m_t)^2 + (\delta m_b)^2} = \sqrt{(0.1)^2 + (0.03)^2} = 0.104 \text{ kg}
\]

With the correct number of significant figures,

\[
m_a \pm \delta m_a = 1.8 \pm 0.1 \text{ kg}.
\]

If \( q \) is a product or quotient like

\[
q = \frac{xz}{y},
\]

\[
\delta q = |q| \sqrt{\left( \frac{\delta x}{x} \right)^2 + \left( \frac{\delta y}{y} \right)^2 + \left( \frac{\delta z}{z} \right)^2}.
\] (0.3)

Example 2

If a certain object has a mass \( m \) of 50.0 \( \pm \) 0.2 kg and a volume \( V \) of 0.250 \( \pm \) 0.005 m\(^3\), the density \( \rho \) is given by

\[
\rho = \frac{m}{V} = \frac{50 \text{ kg}}{0.250 \text{ m}^3} = 200 \frac{\text{kg}}{\text{m}^3}.
\]

According to Equation 0.3, the uncertainty in the density \( \delta \rho \) is given by

\[
\delta \rho = \rho \sqrt{\left( \frac{\delta m}{m} \right)^2 + \left( \frac{\delta V}{V} \right)^2} = \left( 200 \frac{\text{kg}}{\text{m}^3} \right) \sqrt{\left( \frac{0.2}{50} \right)^2 + \left( \frac{0.005}{0.250} \right)^2} = 4.1 \frac{\text{kg}}{\text{m}^3}.
\]

The density is calculated to be

\[
\rho = 200 \pm 4 \frac{\text{kg}}{\text{m}^3}.
\]

If \( q \) is a power of \( x \) and \( y \) like

\[
q = \frac{zx^m}{y^n},
\]

\[
\delta q = |q| \sqrt{\left( \frac{\delta x}{x} \right)^2 + \left( \frac{m \delta y}{y} \right)^2 + \left( \frac{\delta z}{z} \right)^2}.
\] (0.4)

If \( q \) is related to \( x \) by an exactly-known coefficient \( B \) then

\[
q = Bx,
\]

\[
\delta q = |B| \delta x.
\] (0.5)
**KEY IDEAS**

1. When uncertain measured values are used in calculations, *error propagation* is the set of tools used to determine the uncertainty in the calculated result.

**GAME ROUND 2: CALCULATED UNCERTAINTIES**

**AREA OF A METAL BLOCK**

**Submission A1:** Using the vernier calipers, measure, in mm, the two longest sides of the metal block and calculate its area (mm$^2$), and the uncertainty in the area calculation. Submit your measurement and its uncertainty on a half sheet of paper. The submission should include your calculation of the area, the calculation of the uncertainty (with all details), and the payout amount.

**PERIOD OF A TURNTABLE**

**Submission T1:** Using the stopwatch, measure, in seconds, the amount of time it takes for the turntable to make one rotation. Measure the duration of one rotation directly, stopping the watch after exactly one rotation. Note: although the stopwatch measures in increments of 0.01 s, the uncertainty in this measurement is much greater than 0.01 s, and is limited by the inconsistencies in human reaction time. With your group, discuss the best approach for estimating the uncertainty in the time measurement process. Submit your measurement and its uncertainty on a post-it note. The submission should include your measurement of the period, estimate of uncertainty, and the payout amount.

**Submission T2:** If we measure the duration of several rotations of the turntable, we can significantly reduce the uncertainty in the duration of one rotation. Use this technique, along with Equation 0.5 to make a better measurement of the single-rotation duration. Submit your measurement and its uncertainty on a post-it note. The submission should include your measurement of the period, estimate of uncertainty, and the payout amount.

**REPEATED MEASUREMENTS**

When measuring the period of the turntable above, we identified the greatest source of uncertainty as the reaction-time of the experimenter. Even a careful experimenter is just as likely to overshoot her measurement as undershoot. Reaction-time is a source of random error in an experiment. There are many sources of random errors in any particular measurement.

Suppose we need to measure some quantity $x$ and have determined the systematic errors to be negligible. The remaining sources of uncertainty are random, so we should be able to detect them by repeating the measurement several times. Suppose we repeat the measurement five times, and record the (unitless) values

\[45, 46, 46, 49, 43\]

What should we take for our best estimate $x_{\text{best}}$ of the quantity $x$?

Reasonably, we can use the average or mean $\bar{x}$ of the five values as an estimate of $x_{\text{best}}$

\[x_{\text{best}} = \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i\]  

(0.6)
For the five values above, we find

\[ x_{\text{best}} = \bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i = \frac{45 + 46 + 46 + 49 + 43}{5} = 45.8. \]

By taking several measurements we can also get a sense of the uncertainty in our experiment. The standard deviation

\[ \sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2} \]

is a measure that is used to quantify the amount of variation in a set of data. Data sets with large variations have a larger uncertainty than those with small variations. \( \sigma \) measures how much uncertainty is in any one data point \( x_i \).

But, when we combine several \( (N) \) measurements \( x_i \) to make an improved estimate of the measurement \( x_{\text{best}} = \bar{x} \), we expect \( x_{\text{best}} \) to be closer to the true value than one measurement taken alone.

When several individual measurements are combined as an average, we estimate the uncertainty using the standard deviation of the mean, given by

\[ \sigma_{\bar{x}} = \frac{\sigma}{\sqrt{N}} = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} (x_i - \bar{x})^2} \quad (0.7) \]

Note the factor of \( \sqrt{N} \) in the denominator: If we collect more data points, the uncertainty is reduced.

For the five data points shown above the uncertainty is given by

\[ \sigma_{\bar{x}} = \sqrt{\frac{1}{N(N-1)} \sum_{i=1}^{N} (x_i - \bar{x})^2} = \sqrt{\frac{1}{5(5-1)} \left( (45 - 45.8)^2 + (46 - 45.8)^2 + (46 - 45.8)^2 + (49 - 45.8)^2 + (43 - 45.8)^2 \right)} = 1.6 \]

The measurement is correctly reported as 45.8 ± 1.6.

**GAME ROUND 3: REPEATED MEASUREMENTS**

**FALLING TIME OF AN ATWOOD MACHINE**

**SUBMISSION P1:** Using a stop watch, measure the amount of time it takes for the left block of the Atwood machine to fall between the two arrows. Start and stop the timer when the bottom edge of the block passes the arrow. **Make the measurement only once**, and submit your best measurement for the fall time and its uncertainty. Submit your measurement and its uncertainty on a post-it note. The submission should include your measurement of the period, estimate of uncertainty, and the payout amount.

**SUBMISSION P2:** Using a stop watch, measure the amount of time it takes for the left block of the Atwood machine to fall between the two arrows. Start and stop the timer when the bottom edge of the block passes the arrow. **Make the measurement five times**, and submit your best measurement for the fall time and its uncertainty. Submit your measurement and its uncertainty on a half sheet of paper. The submission should include your calculation of the area, the calculation of the uncertainty (with all details), and the payout amount.
**Submission P3:** Using a stop watch, measure the amount of time it takes for the left block of the Atwood machine to fall between the two arrows. Start and stop the timer when the bottom edge of the block passes the arrow. **Make the measurement twenty times**, and submit your best measurement for the fall time and its uncertainty. Submit your measurement and its uncertainty on a half sheet of paper. The submission should include your calculation of the area, the calculation of the uncertainty (with all details), and the payout amount.

**Falling Time of a Dry-Ice Puck on a Track**

Newton’s laws predict that the amount of time it should take for the dry ice puck to travel between the arrows is

\[ t = \frac{d}{\sqrt{2gh}} \]  

(0.8)

with \( g = 9.80665 \pm 0.000005 \) m/s². Carefully measure the distance \( d \) and the height \( h \), with their uncertainties and calculate the predicted time \( t \) that the puck will travel between the arrows.

**Submission D1:** Using a stop watch, measure the amount of time it takes for dry ice puck to travel the distance between the two arrows. Start and stop the timer when the right edge of the puck passes the arrow. **Make the measurement five times**, and submit your best measurement for the time and its uncertainty. Is it in agreement with your calculated prediction? Submit your measurement and its uncertainty on a half sheet of paper. The submission should include your calculation of the time, the calculation of the uncertainty (with all details), the measured time (and uncertainty calculation) and the payout amount.

**Key Ideas**

1. Measurements that can be repeated should be.
2. By taking repeated measurements, we can estimate the uncertainty in the measured value.
3. With more repeated measurements, we reduce the uncertainty.
4. We use statistical analysis to estimate the uncertainty of repeated measurements
5. Error propagation and statistical analysis are distinct tools, and both are important.