The Corrected Diffusion Approximation

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What I do

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Cancer!
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- Results
Tissue Structure

a. Epithelial Layer
b. Stromal Layer
c. Smooth Muscle Layer
Goal: Model diffuse reflectance measurements of backscattered light by a turbid medium close to the source.
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- Use a thin continuous beam incident normally on the medium
- Represent medium by a semi-infinite half space
- Given constant scattering and absorption coefficients
Microscopic

Maxwell’s equations provide a rigorous model for EM wave propagation
Light Propagation In Tissue

- **Microscopic**
  Maxwell’s equations provide a rigorous model for EM wave propagation

- **Mesoscopic**
  The Radiative Transport equation provides a model for light propagation as transport of particles
The Radiative Transport Equation is given by:

\[ \hat{s} \cdot \nabla I + \mu_a I + \mu_s \mathcal{L} I = 0 \]
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\[ \mathcal{L} I = I - \int_{S^2} p(\hat{s} \cdot \hat{s}') I(\hat{s}', r) d\hat{s}' \]

\( p \) defines the fraction of power scattered in direction \( \hat{s} \) incident from direction \( \hat{s}' \).

\[ \int_{S^2} p(\hat{s} \cdot \hat{s}') d\hat{s}' = 1 \]
Boundary Condition

For a normally incident, Gaussian beam we consider

\[ I(\mu, \varphi, x, y, 0) - r(\mu)I(-\mu, \varphi, x, y, 0) = \frac{\delta(\mu - 1)}{2\pi} f(x, y), \quad 0 < \mu \leq 1 \]

where,

\[ f(x, y) = \frac{1}{2\pi w^2} e^{-\frac{x^2 + y^2}{2w^2}} \]
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where

\[ f(x, y) = \frac{1}{2\pi w^2} e^{-\frac{x^2 + y^2}{2w^2}} \]

and

\[ I \rightarrow 0 \quad \text{as} \quad z \rightarrow \infty \]

In this, \( r(\mu) \) is the Fresnel reflection coefficient at the boundary.
Light Propagation In Tissue

- Microscopic
  Maxwell’s equations provide a rigorous model for EM wave propagation

- Mesoscopic
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- Macroscopic
  The Diffusion Approximation is an approximation to the RTE.
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Diffusion Approximation

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- We assume isotropic scattering ($g = 0$)
- The Diffusion equation is of the form

$$\nabla \cdot (D \nabla \Phi) - \mu_a \Phi = S.$$

In this, $D = \frac{1}{3(\mu_a + \mu_s(1-g))}$, and $S$ is the interior source term.
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Problem: This is known to be invalid close to the source.
Corrected Diffusion Model

Bridges the gap between Diffusion and RTE for tissues close to the boundary
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- Compute Interior Solution (Diffusion, \( \Phi \))
Corrected Diffusion Model

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- Compute Interior Solution (Diffusion, $\Phi$)
- Compute Boundary Layer Solution (RTE, $\Psi$)
Corrected Diffusion Model

Bridges the gap between Diffusion and RTE for tissues close to the boundary

- Compute Interior Solution (Diffusion, $\Phi$)
- Compute Boundary Layer Solution (RTE, $\Psi$)
- Combine results to satisfy original conditions

$$I(x, y, z, \hat{s}) = \Phi(x, y, z) + \Psi(x, y, z, \hat{s})$$
Three length scales in our analysis \( (\ell_s \ll w \ll \ell_a) \)

- Scattering mean free path, \( \ell_s = \frac{1}{\mu_s} \)
- Characteristic absorption length, \( \ell_a = \frac{1}{\mu_a} \)
- Beam width \( w \)
Derivation of CDA: Rescaling

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- Use our length scales to define small parameters \(\alpha\) and \(\beta\)
  \[
  \alpha = \frac{\ell_s}{\ell_a} \\
  \beta = \frac{\ell_s}{w}
  \]
Derivation of CDA: Rescaling

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Use our length scales to define small parameters $\alpha$ and $\beta$

- $\alpha = \frac{\ell_s}{\ell_a}$
- $\beta = \frac{\ell_s}{w}$

Rescale $(x, y, z)$ with respect to $w$ which nondimensionalizes the problem

Solve the rescaled, nondimensionalized equation using the fact that $\alpha \ll \beta \ll 1$
\[
\beta \mu \partial_z I + \beta \sqrt{1 - \mu^2} (\cos \varphi \partial_x I + \sin \varphi \partial_y I) + \alpha I + \mathcal{L} I = 0
\]
\[ \beta \mu \partial_z I + \beta \sqrt{1 - \mu^2} (\cos \varphi \partial_x I + \sin \varphi \partial_y I) + \alpha I + \mathcal{L} I = 0 \]

Subject to boundary conditions

\[
I(\mu, \varphi, x, y, 0) = \frac{\delta(\mu - 1)}{2\pi} f(x, y) + r(\mu) I(-\mu, \varphi, x, y, 0), \quad 0 < \mu \leq 1,
\]

\[ I \to 0 \quad \text{as} \quad z \to \infty. \]
CDA: Rescaled Problem

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\[ I \to 0 \quad \text{as} \quad z \to \infty. \]

In these, \( r(\mu) \) is the Fresnel reflection coefficient at the boundary.

We represent \( I \) as the sum of an interior solution and a boundary layer solution as in [\( \Phi + \Psi \)]

\[ (I = \Phi + \Psi) \]

In solving for $\Phi$, we find that $\phi_0$ must satisfy the nondimensionalized diffusion equation

$$\nabla \cdot (\kappa \nabla \phi_0) - \frac{\alpha}{\beta^2} \phi_0 = 0.$$
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in this $\kappa = \frac{1}{3(1 - g)}$, and we have a solution of the form

$$\Phi = \phi_0(r) - \beta \hat{s} \cdot [3 \kappa \nabla \phi_0(r)] + O(\beta^2),$$
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$$\Phi = \phi_0(r) - \beta \hat{s} \cdot [3\kappa \nabla \phi_0(r)] + O(\beta^2),$$

This solution alone cannot satisfy the boundary condition, and we apply a boundary layer solution
Introduce stretched variable $z = \beta Z$, such that

$$\psi(\hat{s}, x, y, Z) = \Psi(\hat{s}, x, y, \beta Z),$$
Boundary Layer Solution

Introduce stretched variable $z = \beta Z$, such that

$$\psi(\hat{s}, x, y, Z) = \Psi(\hat{s}, x, y, \beta Z),$$

substitute into the RTE

$$\mu \psi_Z + \beta \sqrt{1 - \mu^2}(\cos \varphi \psi_x + \sin \varphi \psi_y) + \alpha \psi + \mathcal{L} \psi = 0$$
We apply the modified boundary condition for $\psi = I - \Phi$

\[
\psi(\mu, \varphi, x, y, 0) - r(\mu)\psi(-\mu, \varphi, x, y, 0) = \frac{\delta(\mu - 1)}{2\pi} f(x, y) - [1 - r(\mu)]\phi_0(x, y, 0) + 3\beta\kappa\mu [1 + r(\mu)]\phi_{0,z}(x, y, 0), \quad 0 < \mu \leq 1
\]

Where $\psi = \psi_0 + \beta\psi_1$
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$$\frac{\delta(\mu - 1)}{2\pi} f(x, y) - [1 - r(\mu)]\phi_0(x, y, 0) + 3\beta\kappa\mu[1 + r(\mu)]\phi_{0,z}(x, y, 0), \quad 0 < \mu \leq 1$$

Where $\psi = \psi_0 + \beta\psi_1$, and $\psi_0$ satisfies the 1-D RTE

$$\mu\psi_{0,Z} + \mathcal{L}\psi_0 = 0.$$ 

$\psi_1$ satisfies

$$\mu\psi_{1,Z} + \mathcal{L}\psi_1 = -\sqrt{1 - \mu^2}(\cos \varphi\psi_{0,x} + \sin \varphi\psi_{0,y})$$
Asymptotic Matching

The 1-D RTE in $\psi$ can be solved as a constant
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We ensure that the constant solution is zero to satisfy $\psi \to 0$ as $Z \to \infty$. 
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This returns the boundary condition for the diffusion approximation:

$$a_0 \phi_0 - b_0 \phi_{0,z} = f_0 f(x, y), \quad z = 0$$
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$a_0$, $b_0$, and $f_0$ are determined numerically using the boundary condition for $\psi$ and a numerically calculated Green’s function for the 1-D RTE
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We next solve for $\phi$ and then apply the full boundary condition with the numerically calculated Green’s function to determine $\psi$
We can now solve

\[ \nabla \cdot (\kappa \nabla \phi) - \alpha \phi = 0, \]
\[ a_0 \phi - b_0 \phi_z = f_0 f(xy), \quad \text{at} \quad z = 0. \]
We can now solve
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Using Fourier Transforms \((x, y) \rightarrow (\xi, \eta)\)

\[ -\xi^2 \kappa \hat{\phi} - \eta^2 \kappa \hat{\phi} + \kappa \partial_z^2 \hat{\phi} - \alpha \hat{\phi} = 0, \]
We can now solve

\[ \nabla \cdot (\kappa \nabla \phi) - \alpha \phi = 0, \]

\[ a_0 \phi - b_0 \phi_z = f_0 f(xy), \quad \text{at} \quad z = 0. \]

Using Fourier Transforms \((x, y) \rightarrow (\xi, \eta)\)

\[ -\xi^2 \kappa \hat{\phi} - \eta^2 \kappa \hat{\phi} + \kappa \partial_z^2 \hat{\phi} - \alpha \hat{\phi} = 0, \]

Since \(\phi\) decays exponentially in \(z\) we set \(\gamma(\xi, \eta) = -\sqrt{\alpha/\kappa + \xi^2 + \eta^2}.\)

Substituting this into the BC we find

\[ \hat{\phi} = \frac{f_0 f(\xi, \eta)}{a_0 + b_0 \gamma}, \quad z = 0. \]
Reflectance Calculation

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- Integrate with our source terms to solve for $\Psi$ and $\Phi$, $I = \Psi + \Phi$
Reflectance Calculation

- Solve 1D RTE with Plane Wave Solutions
- Build Greens Function numerically
- Integrate with our source terms to solve for $\Psi$ and $\Phi$, $I = \Psi + \Phi$
- Integrate over the range of angles exiting the medium to determine reflectance at the boundary

$$R(x, y) = -\int\int_{NA} I(r, \hat{s})\hat{s} \cdot \hat{z} d\hat{s}.$$  

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Results: How good is it?

\[ \mu_a = 0.2 (mm)^{-1}, \mu_s = 100 (mm)^{-1}, g = 0.8, n_{rel} = 1.4, \text{ BeamFWHM} = 1 \]
Results: How good is it?

\[ \mu_a = 2 (mm)^{-1}, \mu_s = 100 (mm)^{-1}, g = 0.8, n_{rel} = 1.4, \text{BeamFWHM} = 1 \]
Results: How good is it?

\[ \mu_a = 5 (mm)^{-1}, \mu_s = 100 (mm)^{-1}, g = 0.8, n_{rel} = 1.4, BeamFWHM = 1 \]
$\mu_a = 10 (mm)^{-1}$, $\mu_s = 100 (mm)^{-1}$, $g = 0.8$, $n_{rel} = 1.4$, $Beam FWHM = 1$
Conclusions and Acknowledgements

We constructed a forward model for accurate reflectance measurements close to the source

- We have extended it to include Fresnel reflection, layered tissues, and oblique incidence
- These models give us an option for modeling epithelial tissue specifically in an effort to locate early stage cancer cells, as well as an effective and invertible model for calculating optical properties of tissue
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To Do:

- Inverse problem
- Spatial frequency domain problem
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