Section 3 - Motion and the Calculus

Section Outline
1. The Calculus of Motion
2. The Special Case of Constant Acceleration

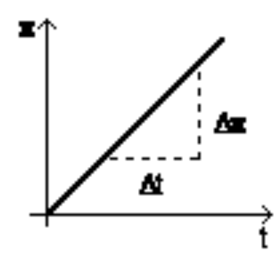
We are trying to answer the question, "What do objects do?" That is, we want to thoroughly describe motion in terms of position, displacement, velocity and acceleration and we have carefully defined these ideas.

Quantity	Definition	Mathematical Representation
Position	The location of an object with respect to a coordinate system.	x
Displacement	A change in position.	$\Delta x \equiv x_f - x_i$
Average Velocity	The average rate of displacement.	$\bar{v} \equiv \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$
Speed	The magnitude of the velocity.	
Average Acceleration	The rate of change of velocity.	$\bar{a} \equiv \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$

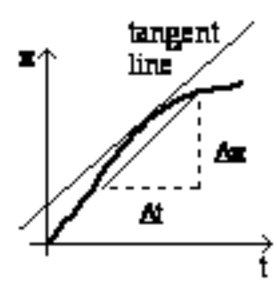
Our understanding of the meaning of these definitions seems involved with slopes and areas under curves. These are the central ideas of the calculus. Let's try to merge our understandings of the definitions with our knowledge of calculus.

1. The Calculus of Motion

If the graph of position versus time is a straight line, then definition of average velocity can be interpreted as the slope of the graph, as shown at the right.



If the graph of position versus time is curved, then the average velocity (slope) will change depending on the size of Δt. You could imagine taking smaller and smaller Δt's and eventually, you will get the slope of the tangent line. The slope of this line is the velocity at that instant. The instantaneous velocity is then,

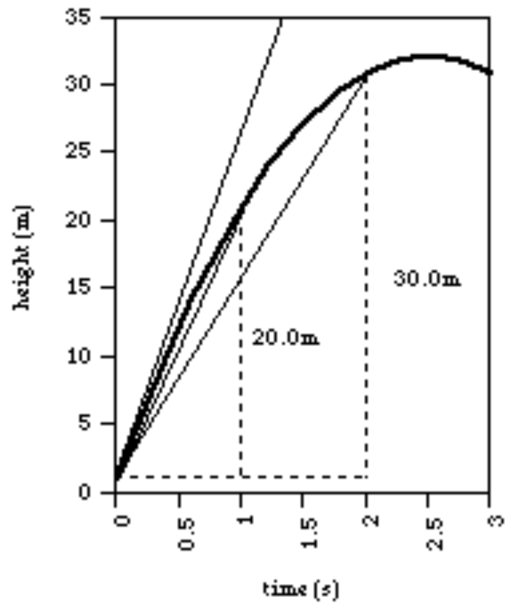


$$v = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}.$$

This is the very definition of the mathematical concept of a derivative. So we can now define,

The Definition of Instantaneous Velocity: $v \equiv \frac{dx}{dt}$

Example 3.1: The position of a ball tossed upward is given by the equation $y = 1.0 + 25t - 5.0t^2$. *(a)Sketch the graph of position versus time. Find (b)the average velocity for the first 2.00s, (c)the average velocity for the first 1.00s, (d)the instantaneous velocity as a function of time, and (e)the velocity at t=0s.*



Given: $y = 1.00 + 25.0t - 5.00t^2$
Find: $\bar{v}$ = ? , v(t) = ?, and v = ?

(b)Using the definition of average velocity and getting the data from the graph,
$$\bar{v} \equiv \frac{\Delta y}{\Delta t} = \frac{y_f - y_i}{t_f - t_i} = \frac{31-1}{2-0} \Rightarrow \boxed{\bar{v} = 15.0 m/s}.$$
(c)In the same manner as part (b),
$$\bar{v} \equiv \frac{\Delta y}{\Delta t} = \frac{y_f - y_i}{t_f - t_i} = \frac{21-1}{1-0} \Rightarrow \boxed{\bar{v} = 20.0 m/s}.$$
(d)Using the definition of instantaneous velocity,
$$v \equiv \frac{dy}{dt} = \frac{d}{dt}\left(1.0 + 25t - 5.0t^2\right) = 25 - 10t .$$
(e)Using the result of part (d) and use t = 0s,
$$v = 25.0 - 10.0(0) \Rightarrow \boxed{v = 25.0 m/s}.$$

Notice that the average velocities tend toward the instantaneous velocity as the Δt gets smaller and smaller, as expected.

As with the change from average velocity to instantaneous velocity, we can go from average acceleration to instantaneous acceleration by using smaller and smaller Δt's eventually leading to infinitesimal dt's and the calculus,

The Definition of Instantaneous Acceleration $a \equiv \frac{dv}{dt}$

In the same way that velocity can be interpreted as the slope of the position versus time graph, the acceleration is the slope of the velocity versus time curve.

Example 3.2: The position of a ball tossed upward is given by the equation $y = 1.0 + 25t - 5.0t^2$. *Find the acceleration of the ball as a function of time.*

Given: $y = 1.0 + 25t - 5.0t^2$
Find: a(t) = ?

Using the definition of instantaneous acceleration and instantaneous velocity,
$$a \equiv \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d}{dt}\left(\frac{d}{dt}(1.0 + 25t - 5.0t^2)\right).$$
Taking the first derivative,
$$a = \frac{d}{dt}\left(25 - 2(5.0)t\right) = \frac{d}{dt}\left(25 - 10t\right).$$

Completing the second derivative,

$$a = \frac{d}{dt}(25 - 10t) \Rightarrow a = -10\,\tfrac{m}{s^2}$$

2. The Special Case of Constant Acceleration

Everything necessary to describe the motion of any object is contained in the definitions of velocity and acceleration,

$$v \equiv \frac{dx}{dt} \text{ and } a \equiv \frac{dv}{dt}.$$

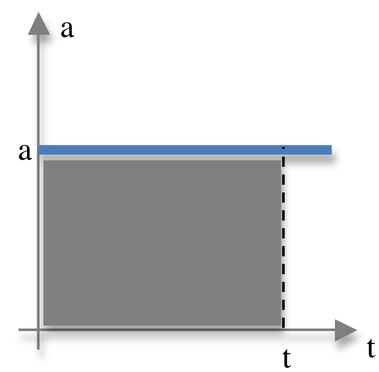
Furthermore, once we know the acceleration as a function of time, the initial velocity, and the initial position we can completely describe all future motion. To illustrate this idea, lets apply these definitions to the special case of constant acceleration and apply our knowledge of calculus.

First, rewrite the definition of acceleration,

$$a \equiv \frac{dv}{dt} \Rightarrow dv = adt\ .$$

Now integrate both sides. The limits can be found by assuming the velocity starts at v_o when t = 0 and ends at a time, t, when the velocity is v,

$$\int_{v_o}^{v} dv = \int_{0}^{t} a\,dt\ .$$



The left side is just the change in the velocity,

$$\int_{v_o}^{v} dv = \int_{0}^{t} a\,dt \Rightarrow v\Big|_{v_o}^{v} = \int_{0}^{t} a\,dt \Rightarrow v - v_o = \int_{0}^{t} a\,dt\ .$$

So the change in velocity is equal to the area under the acceleration versus time graph. Since the acceleration is constant, the graph is a flat line as shown at the right. The area of the region we need is shaded and equal to the product of the acceleration and the time. Let's do the calculus now and see that we get the same answer.

Since the acceleration is constant, it can come through the integral sign. Integrating dt, just gives t evaluated at zero and the final time.

$$v - v_o = \int_{0}^{t} a\,dt \Rightarrow v - v_o = a\int_{0}^{t} dt \Rightarrow v - v_o = at\Big|_{0}^{t} \Rightarrow v - v_o = a(t-0) \Rightarrow v - v_o = at\ .$$

So, in fact, we get the same result as we did looking at the graph. Solving for the final speed,

$$v = v_o + at\ .$$

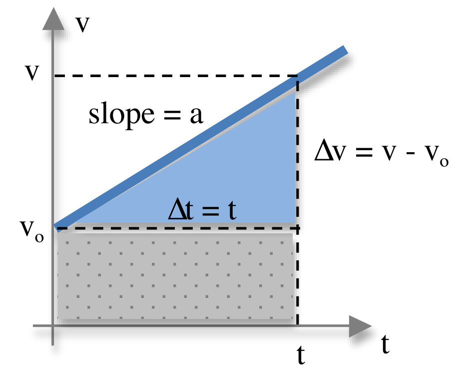
This is the equation for the velocity as a function of time or, in other words, if we know the acceleration and the initial speed, we can find the velocity at any future time.

Now let's find the position as a function of time starting with the definition of velocity and integrating,

$$v \equiv \frac{dx}{dt} \Rightarrow dx = vdt \Rightarrow \int_{x_o}^{x} dx = \int_{0}^{t} v\,dt\ .$$

Again, the limits can be found by assuming the position starts at x_o when t = 0 and ends at a time, t, when the position is x. The left side is just the change in the position,

$$\int_{x_o}^{x} dx = \int_{0}^{t} v\,dt \Rightarrow x\Big|_{x_o}^{x} = \int_{0}^{t} v\,dt \Rightarrow x - x_o = \int_{0}^{t} v\,dt\ .$$



So, the change in position is equal to the area under the velocity time curve. We just figured out how the velocity depends upon time and got, $v = v_o + at$, which is a straight line with a slope of a and a y-intercept of v_o as shown at the right.

The area under the curve is the area of the gray rectangle plus the area of the blue triangle. The area of the rectangle is, $A_r = v_o t$. The area of the triangle is one-half the base times height, $A_t = \frac{1}{2}\Delta v \Delta t$. Since the slope is a,

$a = \dfrac{\Delta v}{\Delta t} \Rightarrow \Delta v = a\Delta t = at$. So, the area of the triangle is,

$A_t = \frac{1}{2}\Delta v \Delta t = \frac{1}{2} a \Delta t \Delta t = \frac{1}{2} a(\Delta t)^2 = \frac{1}{2}at^2$, and the total area is,

$A = v_o t + \frac{1}{2}at^2$.

Now that we know the area, let's do the calculus and see that we get the same answer. Recall we had,

$$x - x_o = \int_0^t v\,dt \,.$$

Substituting the expression for the velocity as a function of time from above,

$$x - x_o = \int_0^t (v_o + at)\,dt = \int_0^t v_o\,dt + \int_0^t at\,dt \,.$$

Since the acceleration and initial velocity don't change with time,

$$\int_{x_o}^{x} dx = v_o \int_0^t dt + a\int_0^t t\,dt \Rightarrow x - x_o = v_o t\Big|_0^t + \tfrac{1}{2}at^2\Big|_0^t \Rightarrow x - x_o = v_o t + \tfrac{1}{2}at^2 \,.$$

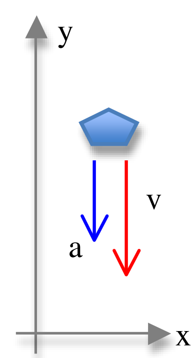
This agrees with the area we calculated earlier. Solving for the final position,

$$x = x_o + v_o t + \tfrac{1}{2}at^2 \,.$$

So, if the know the acceleration, the initial position, and the initial speed we can find the position at any future time.

This method of finding the equations of motion of an object starting from the definitions of velocity and acceleration will work for any non-constant acceleration as well. The only difference is that the integrations are more difficult. We are in effect, able to predict the motion of any object if we can figure out the equation for the acceleration as a function of time.

Example 3.3: A stone is dropped from a height of 30.0m at t = 0s. The acceleration as it falls is $9.80 m/s^2$. Find the equation for (a)the velocity of the stone at any time and (b)the height of the stone at any time.



Given: y_o = 30.0m, v_o = 0, and a = $-9.80 m/s^2$.
Find: v(t) = ? and y(t) = ?

(a)Using the definition of instantaneous acceleration,

$$a \equiv \frac{dv}{dt} \Rightarrow dv = a\,dt \,.$$

Now integrate both sides,

$$\int_{v_o}^{v} dv = \int_0^t a\,dt \Rightarrow v - v_o = a\int_0^t dt \Rightarrow v - v_o = at \Rightarrow v = v_o + at \ .$$

Finally plugging in the numbers,

$$v = 0 + (-9.80\tfrac{m}{s^2})t \Rightarrow \boxed{v = (-9.80\tfrac{m}{s^2})t}.$$

(b) Using the definition of instantaneous velocity,

$$v \equiv \frac{dy}{dt} \Rightarrow dy = vdt = (v_o + at)dt \ ,$$

where we have substituted the equation for velocity as a function of time from part (a). Integrating both side,

$$\int_{y_o}^{y} dy = \int_0^t (v_o + at)\,dt \Rightarrow y - y_o = \int_0^t v_o\,dt + \int_0^t at\,dt \ .$$

Since the acceleration and initial velocity are constants,

$$y - y_o = v_o\int_0^t dt + a\int_0^t t\,dt \Rightarrow y - y_o = v_o t + \tfrac{1}{2}at^2 \ .$$

Solving for the position and plugging in the given values,

$$y = y_o + v_o t + \tfrac{1}{2}at^2 = (30.0m) + (0)t + \tfrac{1}{2}(-9.80\tfrac{m}{s^2})t^2 \Rightarrow \boxed{y = (30.0m) + (-4.90\tfrac{m}{s^2})t^2}.$$

We can check the answer by differentiating twice to get the velocity and then the acceleration,

$$v \equiv \frac{dy}{dt} \Rightarrow v = \frac{d}{dt}\left[(30.0m) + (-4.90\tfrac{m}{s^2})t^2\right] = 0 + (-4.90\tfrac{m}{s^2})\frac{d}{dt}t^2 = (-4.90\tfrac{m}{s^2})2t \Rightarrow v = (-9.80\tfrac{m}{s^2})t \ .$$

and

$$a \equiv \frac{dv}{dt} \Rightarrow a = \frac{d}{dt}\left[(-9.80\tfrac{m}{s^2})t\right] = (-9.80\tfrac{m}{s^2})\frac{d}{dt}t \Rightarrow a = (-9.80\tfrac{m}{s^2}) ,$$

in agreement with the given acceleration.

Section 3 - Summary

Our goal is to understand what objects do and why they do it. The point of this section is that we can use calculus to add to our understanding of what objects do. That is, we can describe their motion in terms of position, displacement, velocity and acceleration. We have carefully defined these ideas and built a complete back story to understand these concepts in various ways including in terms of the calculus.

Quantity	Definition	Mathematically
Position	The location of an object with respect to a coordinate system.	x
Displacement	A change in position.	dx
Velocity	The rate of displacement or the rate of change of position.	$v \equiv \frac{dx}{dt}$
Acceleration	The rate of change of velocity.	$a \equiv \frac{dv}{dt}$
Speed	The magnitude of the velocity.	$\lvert v\rvert$