Section 4 - Constant Acceleration & Freefall

Section Outline
1. The Special Case of Constant Acceleration
2. Freely Falling Bodies

As we continue to develop our answer to the question, "What do objects do?" we need to apply our general understanding of motion to a specific case. We'll choose the motion due to constant acceleration, just as an example. We we'll use the basic definitions we have built to develop the "equations of motion" for constant acceleration. These equations are often called the "kinematic equations."

1. The Special Case of Constant Acceleration

The Scientific Method requires that our theories are predictive. Our theory of motion must be able to tell us the future motion of any object. If we know where an object starts, how fast it is initially going and its acceleration at every time, we need to be able to predict where it will be and how fast it will be going at any future time. Mathematically, this problem is stated as follows: Given the initial position is x_o, the initial velocity is v_o, and the acceleration as a function of time a(t), find the velocity as a function of time v(t), the position as a function of time x(t), the acceleration as a function of position a(x), and the velocity as a function of position v(x). Simply put, we want the equations that completely describe the motion of the object – the "equations of motion."

In general, this can be hard to do, so let's start by looking at the least complicated situation where the acceleration is constant. We could do this using the calculus, but let's use algebra just so that we feel a bit more comfortable. The equations of motion we will get are called often the "Kinematic Equations."

Since the acceleration is constant, $\boxed{a(t) = a}$ and $\boxed{a(x) = a}$ where a is the value of the constant acceleration. To find the velocity as a function of time, start with the definition of average acceleration,

$$\bar{a} \equiv \frac{\Delta v}{\Delta t} \Rightarrow a = \frac{v(t) - v_o}{t - 0} \Rightarrow \boxed{v(t) = v_o + at}.$$

This is the velocity as a function of time and it agrees with our earlier result using the calculus.

To find the position as a function of time, use the definition as average velocity,

$$\bar{v} \equiv \frac{\Delta x}{\Delta t} \Rightarrow \tfrac{1}{2}(v + v_o) = \frac{x(t) - x_o}{t - 0} \Rightarrow x(t) - x_o = \tfrac{1}{2}(v + v_o)t.$$

Substituting the velocity as a function of time,

$$x(t) - x_o = \tfrac{1}{2}(v_o + at + v_o)t = v_o t + \tfrac{1}{2}at^2 \Rightarrow \boxed{x(t) = x_o + v_o t + \tfrac{1}{2}at^2}.$$

This is the position as a function of time and it also matches the calculus result.

To get the velocity as a function of position, solve the equation for velocity as a function of time for the time and substitute into the equation for position as a function of time,

$$v(t) = v_o + at \Rightarrow t = \frac{v - v_o}{a} \Rightarrow x = x_o + v_o\left(\frac{v - v_o}{a}\right) + \tfrac{1}{2}a\left(\frac{v - v_o}{a}\right)^2.$$

Now solve for v,

$$x = x_o + v_o\left(\frac{v - v_o}{a}\right) + \tfrac{1}{2}a\left(\frac{v - v_o}{a}\right)^2 = x_o + \frac{vv_o - v_o^2}{a} + \frac{v^2 - 2vv_o + v_o^2}{2a}$$

$$x = x_o + \frac{2vv_o - 2v_o^2 + v^2 - 2vv_o + v_o^2}{2a} = x_o + \frac{v^2 - v_o^2}{2a} \Rightarrow \boxed{v^2(x) = v_o^2 + 2a(x - x_o)}.$$

Given the initial position x_o, the initial velocity v_o, and the constant acceleration a, we can use the equations we just created to predict the acceleration at any time, the velocity at any time, the position at any time, the acceleration at any position and the velocity at any position. Here's a summary:

$$a(t) = a$$
$$v(t) = v_o + at$$
$$x(t) = x_o + v_o t + \tfrac{1}{2}at^2$$
$$a(x) = a$$
$$v^2(x) = v_o^2 + 2a(x - x_o)$$

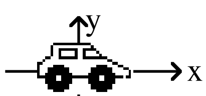
The motion of an object undergoing constant acceleration is completely described!

Since sometimes you aren't given the initial position or initial velocity, it is convenient to rearrange these equations into a more suitable set commonly called the kinematic equations.

Kinematic Equations	
$x = x_o + v_o t + \frac{1}{2}at^2$	missing v
$v = v_o + at$	missing x
$v^2 = v_o^2 + 2a(x - x_o)$	missing t
$x - x_o = \frac{1}{2}(v_o + v)t$	missing a

There are six quantities that appear in these equations, x, x_o, v, v_o, a, and t. Since x_o is determined by your choice of coordinate system, there are really just five variables. Each of these four equations is missing just one of the five variables. The kinematic equations are ideal for problem solving because if you are given any combination of three of the variables, you can solve for the remaining two unknowns by choosing the equation that has only one of the unknowns and solving for it. Then you can use a second equation to solve the remaining unknown.

Example 4.1: A car accelerating at a constant rate goes from zero to 60mph (26.8m/s) in 6.00s. Find (a)the acceleration and (b)the distance traveled during this time.



Given/Find:
$x_o = 0$
$x = ?$
$v_o = 0$
$v = 26.8$m/s
$a = ?$
$t = 6.00$s

The first thing we do is chose a coordinate system. The most convenient one has x_o=0 at t=0. Now we list the knowns and unknowns.

(a)Using the kinematic equation without the final position,

$$v = v_o + at \Rightarrow a = \frac{v}{t} = \frac{26.8}{6} \Rightarrow \boxed{a = 4.47\text{m/s}^2}$$

(b)Using the kinematic equation without the acceleration,

$$x - x_o = \tfrac{1}{2}(v + v_o)t \Rightarrow x = \tfrac{1}{2}vt = \tfrac{1}{2}(26.8)(6.00) \Rightarrow \boxed{x = 80.4\text{m}}$$

COMMENT ON PROBLEM SOLVING:
First, notice that it is critical to indicate the coordinate system because the value of x_o depends upon this choice. You must always indicate your coordinate system in your sketch of the problem. Second, in these Kinematic problems it is convenient to start by listing the six quantities, then fill in the knowns stated in the problem. The unknowns that remain help decide which Kinematic Equation is needed. Finally, you should always do the algebra first to solve the quantity you seek, then plug the numbers into the resulting equation.

2. Freely Falling Bodies

When air resistance is so small that it can be ignored, a ball and feather will fall at the same rate. Check out this YouTube (http://www.youtube.com/watch?v=ndFXXasM6ZE). This experiment was even carried out on the moon with a hammer and a feather! See it at the NASA web site (http://nssdc.gsfc.nasa.gov/planetary/lunar/apollo_15_feather_drop.html). This leads us to…

Galileo's Rule of Falling Bodies:

All objects near the surface of Earth, regardless of their mass, fall toward Earth with an acceleration of $9.80 m/s^2$ when air resistance is small enough to ignore.

It is really amazing that mass doesn't matter. The two YouTube videos show experimental evidence that the mass doesn't matter, yet most people still find it disturbing. We'll work on understanding it better when we talk about the causes of motion. So please suspend your worries for a bit, accept the experimental evidence, and we'll address your very valid concerns in a short while.

One more thing to note, we call this a "rule" instead of a "law" because we will actually be able to explain why it is so from a deeper explanation of gravity due to Isaac Newton. Remember that a "law" cannot be proven true, it can only be validated from repeated testing. Galileo's Rule can be proven from Newton's gravitational principles, so Galileo's Rule can't be a law. Having said that, you will often see Galileo's Rule mistakenly referred to as a law in many places.

So, when is air resistance small enough to ignore? In the real world of thrown baseballs and paper airplanes, air resistance is an important factor in the resulting motion. For now, we are really trying hard to just understand the least complex kind of motion, constant acceleration. So, let's agree to assume that all falling objects in the following examples feel no air resistance and undergo a constant acceleration of $9.80 m/s^2$ that we call "freefall." We represent the number $9.80 m/s^2$ with the symbol g.

The Acceleration Due to Gravity: $g = 9.80 m/s^2$

Example 4.2: A ball is thrown upward with a speed of 15.0m/s. Find (a)the maximum altitude and (b)the time it takes to get there.

It looks like there is not enough information to solve this problem since we are only given the initial speed. However, we can get the initial position by choosing the coordinates and we know the freefall acceleration is $9.80 m/s^2$. If the final speed were a positive number the ball would still be going up. If it were a negative number the ball would already be headed down. The final speed at the top of the motion must be zero. Notice that we can use the variable y in the kinematic equations for motion along the y-axis.

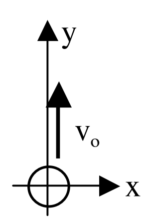
(a)Use the kinematic equation without the time,

$$v^2 = v_o^2 + 2a(y - y_o) \Rightarrow 0 = v_o^2 + 2ay \Rightarrow y = -\frac{v_o^2}{2a}.$$

Plugging in the numbers, $y = -\frac{15^2}{2(-9.8)} \Rightarrow \boxed{y = 11.5\text{m}}$.

(b)Use the kinematic equation without the final position,

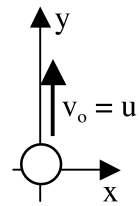
$$v = v_o + at \Rightarrow t = -\frac{v_o}{a} = -\frac{15}{-9.8} \Rightarrow \boxed{t = 1.53\text{s}}.$$



Given/Find:
$y_o = 0$
$y = ?$
$v_o = 15.0 m/s$
$v = 0$
$a = -9.80 m/s^2$
$t = ?$

Example 4.3: A ball is thrown upward with a speed, u. (a)Show that the time it takes to reach maximum height is equal to the time it takes to fall back down. (b)Find the velocity when it gets there.

Let's break this up into two problems, the upward motion and the downward motion.



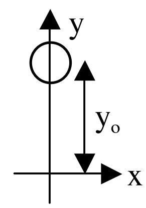
Given/Find:
$y_o = 0$
$y = ?$
$v_o = u$
$v = 0$
$a = -g$
$t = ?$

(a)Use the kinematic equation without the final position,

$$v = v_o + at \Rightarrow 0 = u - gt \Rightarrow \boxed{t_{up} = \frac{u}{g}}.$$

For the downward motion we will need the maximum height, so using the kinematic equation without the time,

$$v^2 = v_o^2 + 2a(y - y_o) \Rightarrow 0 = u^2 - 2gy \Rightarrow y = \frac{u^2}{2g}.$$



Given/Find:
$y_o = \frac{u^2}{2g}$
$y = 0$
$v_o = 0$
$v = ?$
$a = -g$
$t = ?$

For the downward motion, the final height for the ball becomes the initial position and the initial velocity is zero. Using the kinematic equation without the final velocity,

$$y = y_o + v_o t + \tfrac{1}{2}at^2 \Rightarrow 0 = \frac{u^2}{2g} + 0 - \tfrac{1}{2}gt^2 \Rightarrow \tfrac{1}{2}gt^2 = \frac{u^2}{2g} \Rightarrow$$

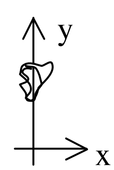
$\boxed{t_{down} = \frac{u}{g}}$. Therefore, $t_{up} = t_{down}$.

(b)Using the kinematic equation for the final speed,

$$v = v_o + at = -gt = -g\left(\frac{u}{g}\right) \Rightarrow \boxed{v = -u}.$$

The final speed is the same as the initial speed.

Example 4.4: A student throws a set of keys vertically up to her roommate at the top of the stairs 4.00m above. The keys are in the air for 1.50s. Find (a)the initial velocity of the keys and (b)the velocity of the keys just before they are caught.



Given/Find:
$y_0 = 0$
$y = 4.00m$
$v_o = ?$
$v = ?$
$a = -9.80m/s^2$
$t = 1.50s$

(a)Using the kinematic equation without the final velocity,

$$y = y_o + v_o t + \tfrac{1}{2}at^2$$

Noting that $y_o = 0$ and solving for the initial velocity,

$$v_o = \frac{y}{t} - \frac{at}{2} = \frac{4.00}{1.50} - \frac{(-9.80)(1.50)}{2} \Rightarrow \boxed{v_o = 10.0m/s}$$

(b)Using the kinematic equation,

$$v = v_o + at = 10.0 + (-9.80)(1.50) \Rightarrow \boxed{v = -4.70m/s}$$

This must mean that the keys were caught on their way down, not on the way up!

Section 4 - Summary

We have shown that if we know the initial position, the initial velocity, and the acceleration as a function of time we can predict everything about the future motion of any object. For the case of constant acceleration we have a set of equations that are ideal for problem solving called the kinematic equations.

Kinematic Equations (These work only for constant acceleration!)

$x = x_o + v_o t + \frac{1}{2}at^2$	missing v
$v = v_o + at$	missing x
$v^2 = v_o^2 + 2a(x - x_o)$	missing t
$x - x_o = \frac{1}{2}(v_o + v)t$	missing a

We established Galileo's Rule of Falling Bodies and defined the acceleration due to gravity.

Galileo's Rule of Falling Bodies: All objects near the surface of Earth, regardless of their mass, fall toward Earth with an acceleration of 9.80m/s^2 when air resistance is small enough to ignore.

The Acceleration Due to Gravity: g = 9.80m/s^2

Where does this leave us? Well, we have to be able to extend our understanding of motion to more than one dimension, which we will do next. There is a deeper issue remaining if we intend to understand what object do and why they do it. We need to know what determines the acceleration of an object. We'll attack this in a short while.