Section 5 – Review of Vectors

Section Outline
1. Vectors
2. Components of a Vector
3. Addition of Vectors
4. Multiplication of Vectors

Vectors are a mathematical tool. Like all math tools, they seem unnecessarily difficult and not too helpful at first. However, once they have been mastered, they turn out to be a great convenience.

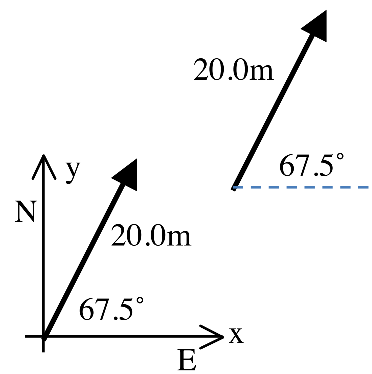
1. Vectors

Scalar: A quantity with a magnitude only.
Vector: A quantity with magnitude and direction.

Example 5.1: A hiker walks 20.0m NNE. Sketch this vector.

The vector as shown at the right. The length of the vector is proportional to the magnitude (20.0m) and the arrow indicates the direction.

Note that vectors can be moved with respect to the origin as long as neither the magnitude nor direction is changed.

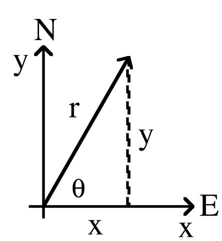


A way to right this vector is, $\vec{r} = 20.0\,\text{m}\ @67.5^\circ$ N of E. This method is cumbersome. We will learn a better way soon. Note: Book often use bold type face for vectors ($\mathbf{r} = \vec{r}$).

2. Components of a Vector

Example 5.2: A hiker walks 20.0m NNE. Find the distance she has gone northward and the distance eastward.

Given:	Find:
$r = 20.0\text{m}$	$x = ?$
$\theta = 67.5^\circ$	$y = ?$



Convert these polar coordinates to Cartesian,

$y = r\sin\theta = 20.0\sin 67.5^\circ \Rightarrow \boxed{y = 18.5\text{m northward}}$ and

$x = r\cos\theta = 20.0\cos 67.5^\circ \Rightarrow \boxed{x = 7.65\text{m eastward}}$.

These parts of the vector are called "components."

Unit Vectors: $\hat{i} \equiv 1$ unit along the x-axis, $\hat{j} \equiv 1$ unit along the y-axis and $\hat{k} \equiv 1$ unit along the z-axis.

Now the answer previous example can be written as, $\vec{r} = (7.65\text{m})\hat{i} + (18.5\text{m})\hat{j}$. This is standard notation because it is very convenient for vector addition and multiplication.

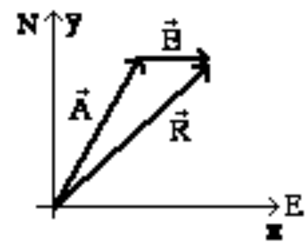
In general vectors are written as, $\vec{r} = r_x\hat{i} + r_y\hat{j} + r_z\hat{k}$, where r_x is the x-component of $\vec{r}$, r_y is the y-component of $\vec{r}$ and r_z is the z-component of $\vec{r}$.

3. Addition of Vectors

The vector $\vec{r}$ can be considered the sum of its components. This suggests that the way to add two arbitrary vectors is to place them "head-to-tail."

Example 5.3: Find the total displacement of a hiker that walks 20.0m NNE then 5.00m due east.

Given:	Find:
$\vec{A} = (7.65m)\hat{i} + (18.5m)\hat{j}$	R = ?
$\vec{B} = (5.00m)\hat{i}$	θ = ?



The answer is the vector $\vec{R}$, which is called the "resultant."

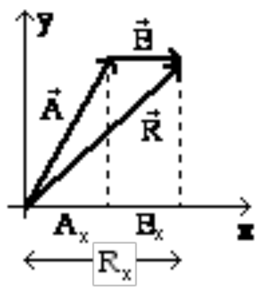
Graphical Method: Draw the vectors to scale and measure.

Analytic Method: Note that the x-component of $\vec{R}$ is equal to the sum of the x-components of $\vec{A}$ and $\vec{B}$.

$R_x = A_x + B_x = 7.65 + 5.00 = 12.7\text{m}$

The same is true for the y-components,

$R_y = A_y + B_y = 18.5 + 0 = 18.5\text{m}$.



Unit Vector Method:

Since $\vec{R} = \vec{A} + \vec{B} \Rightarrow \vec{R} = (A_x\hat{i} + A_y\hat{j}) + (B_x\hat{i} + B_y\hat{j})$.

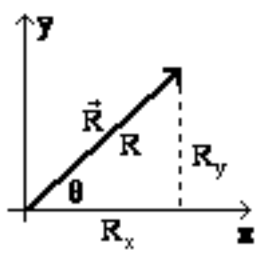
Rearranging terms, $\vec{R} = (A_x + B_x)\hat{i} + (A_y + B_y)\hat{j} = 12.7\hat{i} + 18.5\hat{j}$.

The beauty of the unit vector notation is that you can use regular algebra to add vectors.

Now use the Pythagorean Theorem and the definition of tangent to reconstruct the resultant,

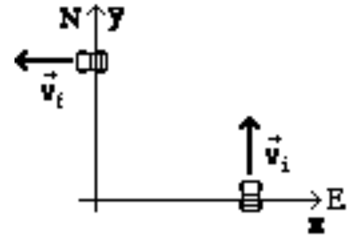
$$R = \sqrt{R_x^2 + R_y^2} = \sqrt{(12.7)^2 + (18.5)^2} \Rightarrow \boxed{R = 22.4\text{m}}$$

$$\theta = \arctan\frac{R_y}{R_x} = \arctan\frac{18.5}{12.7} \Rightarrow \boxed{\theta = 55.5^\circ \text{ North of East}}.$$



Many other quantities should be treated as vectors. In the next example velocity is treated like a vector.

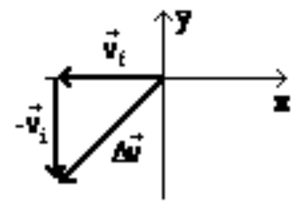
Example 5.4: A car is traveling northward at 75.0km/h. It rounds a curve and is now heading westward at 75.0km/h. Find the change in the velocity of the car.



Given:
$\vec{v}_i = 75.0\hat{j}$
$\vec{v}_f = -75.0\hat{i}$

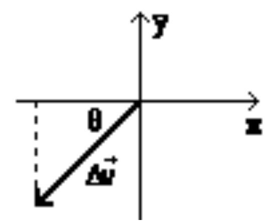
Find:
$\Delta v = ?$
$\theta = ?$

We need $\Delta\vec{v} \equiv \vec{v}_f - \vec{v}_i = \vec{v}_f + (-\vec{v}_i)$.
Vector subtraction is defined to be vector addition of the oppositely directed vector. This is shown in the second sketch.



In unit vector notation, $\Delta\vec{v} = (-75.0\hat{i}) + (-75.0\hat{j}) = -75.0\hat{i} - 75.0\hat{j}$.

For magnitude and direction, use the Pythagorean Theorem and definition of tangent,

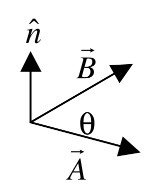


$$\Delta v = \sqrt{\Delta v_x^2 + \Delta v_y^2} = \sqrt{(-75.0)^2 + (-75.0)^2} \Rightarrow \boxed{\Delta v = 106 \text{km/h}}$$

$$\theta = \arctan\frac{\Delta v_y}{\Delta v_x} = \arctan\frac{-75.0}{-75.0} \Rightarrow \boxed{\theta = 45.0^\circ \text{ South of West}}.$$

4. Multiplication of Vectors

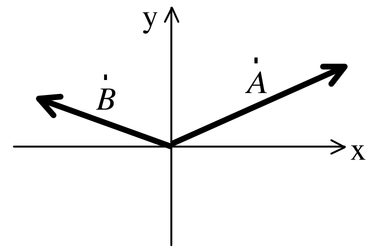
There are two types of vector multiplication. One produces a scalar quantity and is called the "Dot" or "Scalar" product. The second type produces a vector and is called the "Cross" or "Vector" product.



Dot Product: $\vec{A} \bullet \vec{B} \equiv AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$
Cross Product: $\vec{A} \times \vec{B} \equiv AB\sin\theta\hat{n} = (A_yB_z - A_zB_y)\hat{i} + (A_zB_x - A_xB_z)\hat{j} + (A_xB_y - A_yB_z)\hat{k}$
The cross product is perpendicular to both vectors involved and it is often written as,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example 5.5: For the two vectors $\vec{A} = 40.0\hat{i} + 20.0\hat{j}$ *and* $\vec{B} = -30.0\hat{i} + 10.0\hat{j}$ *find (a)their dot product, (b)their cross product and (c)the angle between them.*



Given:
$\vec{A} = 40.0\hat{i} + 20.0\hat{j}$
$\vec{B} = -30.0\hat{i} + 10.0\hat{j}$

Find:
$\vec{A} \bullet \vec{B} = ?$
$\vec{A} \times \vec{B} = ?$

(a)Using the equation for dot product,

$$\vec{A} \bullet \vec{B} = A_x B_x + A_y B_y + A_z B_z = (40)(-30) + (20)(10) \Rightarrow \boxed{\vec{A} \bullet \vec{B} = -1000}.$$

(b)Using the equation for the cross product,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 40 & 20 & 0 \\ -30 & 10 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + [(40)(10) - (20)(-30)]\hat{k} \Rightarrow \boxed{\vec{A} \times \vec{B} = 1000\hat{k}}.$$

The cross product points in the z-direction as it should.

(c)Using the definition of the dot product,

$$\vec{A} \bullet \vec{B} = AB\cos\theta \Rightarrow \theta = \arccos\frac{\vec{A} \bullet \vec{B}}{AB} = \arccos\frac{\vec{A} \bullet \vec{B}}{\sqrt{A_x^2 + A_x^2}\sqrt{B_x^2 + B_x^2}}$$

Plugging in the numbers, $\theta = \arccos\dfrac{-1000}{\sqrt{40^2 + 20^2}\sqrt{(-30)^2 + 10^2}} \Rightarrow \boxed{\theta = 135^\circ}$

<u>Section 5 - Summary</u>

Vectors (notation and properties)

Vector Components $A_x = A\cos\theta$ and $A_y = A\sin\theta$

Unit Vector Notation $\vec{A} = A_x\hat{i} + A_y\hat{j}$

Vector Addition $\vec{R} = \vec{A} + \vec{B}$ where $R_x = A_x + B_x$ and $R_y = A_y + B_y$

Vector Dot Product $\vec{A} \bullet \vec{B} \equiv AB\cos\theta = A_x B_x + A_y B_y + A_z B_z$

Vector Cross Product $\vec{A} \times \vec{B} \equiv AB\sin\theta\hat{n} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_z)\hat{k}$

$$\vec{A} \times \vec{B} \equiv AB\sin\theta\hat{n} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$