Chapter 6 - Describing Motion in More Dimensions

Section Outline
1. The Definition of Position
2. The Definition of Displacement
3. The Definition of Average Velocity
4. The Definition of Speed
5. The Definition of Average Acceleration
6. From Average to Instantaneous

What do objects do? They move in three dimensions not just along straight lines. In this section we will describe motion in three dimensions by extending the same ideas of position, displacement, velocity and acceleration. We won't change their meaning at all, we'll just let them become vectors. This is the first example of the way that physics tries to minimize the number of physical concepts and explain the widest possible variety of phenomena.

Recall, our descriptions of motion in one dimension:

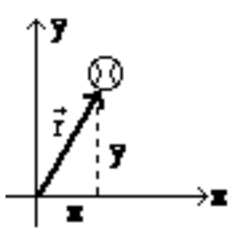
Quantity	Definition	Mathematical Representation
Position	The location of an object with respect to a coordinate system.	x
Displacement	A change in position.	$\Delta x \equiv x_f - x_i$
Average Velocity	The average rate of displacement.	$\bar{v} \equiv \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$
Speed	The magnitude of the velocity.	v = |v|
Average Acceleration	The rate of change of velocity.	$\bar{a} \equiv \frac{\Delta v}{\Delta t} = \frac{v_f - v_i}{t_f - t_i}$

Let's go through each idea again, keeping the definition unchanged, but let the mathematical description become vectorial.

1. The Definition of Position

We'll keep the definition the same:

Position: The location of an object with respect to a coordinate system.

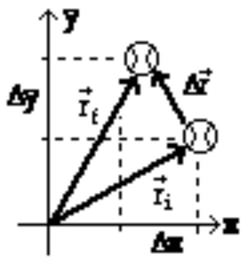


Now we'll let the position become a vector with multiple components. For now we'll just think about two dimensions.

Mathematically, $\vec{r} = x\hat{i} + y\hat{j}$.

<u>2. The Definition of Displacement</u>
Again, we'll keep the definition the same:

Displacement: A change in position.

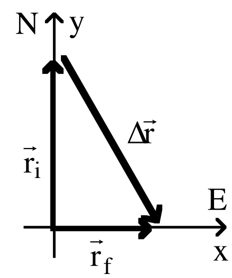


Now we'll let the displacement also become a vector.

Mathematically, $\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = (x_f\hat{i} + y_f\hat{j}) - (x_i\hat{i} + y_i\hat{j}) = (x_f - x_i)\hat{i} + (y_f - y_i)\hat{j}$,

which is more simply written as, $\Delta\vec{r} = \Delta x\hat{i} + \Delta y\hat{j}$.

> *Example 6.1: A hiker walks from a position of 3.00km due north of a ranger station to 1.00km east of the ranger station in 0.500h. Using an origin at the ranger station, find (a)the initial position, (b)the final position, and (c)the displacement of the hiker.*
>
> Given: r_i = 3.00km north an r_f = 1.00km east
> Find: $\vec{r}_i = ?$, $\vec{r}_f = ?$, and $\Delta\vec{r} = ?$
> (a)Using the coordinates shown at the right, $\boxed{\vec{r}_i = 3.00\hat{j}}$ and
> (b) $\boxed{\vec{r}_f = 1.00\hat{i}}$.
> (c)Using the definition of displacement,
> $\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = 1.00\hat{i} - (3.00\hat{j}) \Rightarrow \boxed{\Delta\vec{r} = 1.00\hat{i} - 3.00\hat{j}}$.



<u>3. The Definition of Average Velocity</u>
Keeping the definition the same,

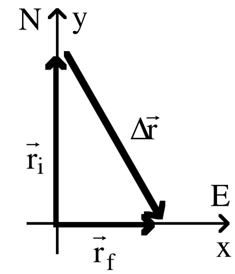
Average Velocity: The average rate of displacement. Mathematically, $\vec{v} \equiv \dfrac{\Delta\vec{r}}{\Delta t}$.

Using the displacement, $\vec{v} = \dfrac{\Delta x\hat{i} + \Delta y\hat{j}}{\Delta t} = \dfrac{\Delta x}{\Delta t}\hat{i} + \dfrac{\Delta y}{\Delta t}\hat{j} = v_x\hat{i} + v_y\hat{j}$.

<u>4. The Definition of Speed</u>
Again, using the old definition,

Speed: The magnitude of the velocity vector. Mathematically, $v \equiv |\vec{v}| = \sqrt{v_x^2 + v_y^2}$

Example 6.2: A hiker walks from a position of 3.00km due north of a ranger station to 1.00km east of the ranger station in 0.500h. Using an origin at the ranger station, find (a)the average velocity and (b)the average speed of the hiker.



Given $\vec{r}_i = 3.00\hat{j}$, $\vec{r}_f = 1.00\hat{i}$, and $\Delta\vec{r} = 1.00\hat{i} - 3.00\hat{j}$.
Find: $\vec{v}$ = ? and v = ?

(a)Using the definition of average velocity,
$$\vec{v} \equiv \frac{\Delta\vec{r}}{\Delta t} = \frac{1.00\hat{i} - 3.00\hat{j}}{0.500} \Rightarrow \boxed{\vec{v} = 2.00\hat{i} - 6.00\hat{j}} \text{ in km/h.}$$
(b)Using the definition of speed,
$$v \equiv |\vec{v}| = \sqrt{v_x^2 + v_y^2} = \sqrt{2^2 + 6^2} \Rightarrow \boxed{v = 6.32\,\text{km/h}}.$$

5. The Definition of Average Acceleration

One last time,

Average Acceleration: The rate of change of velocity. Mathematically, $\vec{a} \equiv \frac{\Delta\vec{v}}{\Delta t}$.

Using the velocity,
$$\vec{a} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t} = \frac{(v_{xf}\hat{i} + v_{yf}\hat{j}) - (v_{xi}\hat{i} + v_{yi}\hat{j})}{\Delta t} = \frac{(v_{xf} - v_{xi})\hat{i} + (v_{yf} - v_{yi})\hat{j}}{\Delta t} = \frac{\Delta v_x}{\Delta t}\hat{i} + \frac{\Delta v_y}{\Delta t}\hat{j} = a_x\hat{i} + a_y\hat{j}.$$

You might be able to guess how to extend from two dimensions to three dimensions. The definitions stay the same, but the vectors now have a z-component.

Position: $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ Displacement: $\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = \Delta x\hat{i} + \Delta y\hat{j} + \Delta z\hat{k}$

Average Velocity: $\vec{v} \equiv \frac{\Delta\vec{r}}{\Delta t} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$ Speed: $v \equiv |\vec{v}| = \sqrt{v_x^2 + v_y^2 + v_z^2}$

Average Acceleration: $\vec{a} \equiv \frac{\Delta\vec{v}}{\Delta t} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$.

Example 3.2: A baseball has a velocity of $35.0\hat{i} + 12.0\hat{j} + 25.0\hat{k}$ *in m/s. Its velocity is* $32.0\hat{i} + 10.0\hat{j} + 5.00\hat{k}$ *after 2.00s. Find (a)the average acceleration vector and (b)the magnitude of the average acceleration.*

Given $\vec{v}_i = 35.0\hat{i} + 12.0\hat{j} + 25.0\hat{k}$, $\vec{v}_f = 32.0\hat{i} + 10.0\hat{j} + 5.00\hat{k}$, and $\Delta t = 2.00$s.
Find: $\vec{a}$ = ? and a = ?

(a)Using the definition of average acceleration,
$$\vec{a} \equiv \frac{\Delta\vec{v}}{\Delta t} = \frac{\vec{v}_f - \vec{v}_i}{\Delta t} = \frac{(32\hat{i} + 10\hat{j} + 5\hat{k}) - (35\hat{i} + 12\hat{j} + 25\hat{k})}{2} \Rightarrow \boxed{\vec{a} = -1.50\hat{i} - 1.00\hat{j} - 10.0\hat{k}} \text{ in m/s}^2.$$
(b)Using the Pythagorean Theorem,
$$a \equiv |\vec{a}| = \sqrt{a_x^2 + a_y^2 + a_z^2} = \sqrt{(-1.5)^2 + (-1)^2 + (-10)^2} \Rightarrow \boxed{a = 10.2\,\text{m/s}^2}.$$

6. From Average to Instantaneous

Just as in the case of one dimensional motion, going from average to instantaneous involves letting the Δt get smaller and smaller. This is the same as taking the limit as Δt goes to zero. The net result, as before, is that the Δ's become d's with all the "rights and privileges" of derivatives. In summary then,

Quantity	Definition	Mathematical Representation
Position	The location of an object with respect to a coordinate system.	$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
Displacement	A change in position.	$d\vec{r} \equiv \vec{r}_f - \vec{r}_i = dx\hat{i} + dy\hat{j} + dz\hat{k}$
Velocity	The average rate of displacement.	$\vec{v} \equiv \frac{d\vec{r}}{dt} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$
Speed	The magnitude of the velocity.	$v \equiv \lvert\vec{v}\rvert = \sqrt{v_x^2 + v_y^2 + v_z^2}$
Acceleration	The rate of change of velocity.	$\vec{a} \equiv \frac{d\vec{v}}{dt} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

Example 6.3: The position of a flying insect as a function of time is given by $\vec{r} = 2\hat{i} - 3t^2\hat{j} + 5t\hat{k}$. *Find (a)the velocity as a function of time and (b)the acceleration as a function of time.*

Given $\vec{r} = 2\hat{i} - 3t^2\hat{j} + 5t\hat{k}$.
Find: $\vec{v}(t) = ?$ and $\vec{a}(t) = ?$.

(a)Using the definition of velocity we just need to take the derivative, $\vec{v} \equiv \frac{d\vec{r}}{dt} \Rightarrow \boxed{\vec{v} = -6t\hat{j} + 5\hat{k}}$.

(a)Using the definition of acceleration in the same way, $\vec{a} \equiv \frac{d\vec{v}}{dt} \Rightarrow \boxed{\vec{v} = -6\hat{j}}$.

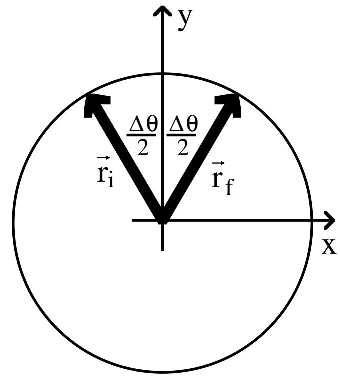
The Small Angle Approximation

The small angle approximation will appear many times in physics. The idea is that as an angle gets smaller and smaller, the sine of the angle gets closer and closer to the value of the angle itself in radians. You can see this happen in the table below.

α (degrees)	α (radians)	sin α
57.3	1.00	0.841
5.73	0.100	0.0998
0.573	0.0100	0.0099998

In summary, the small angle approximation is $\sin\theta \approx \theta$.

Example 6.4: Find the instantaneous velocity of the tip of a 10.0cm long second hand on a clock.



Given: r = 0.100m and T = 60.0s.
Find: $\vec{v} = ?$

Choosing the coordinates shown at the right, the position of the tip of the second hand a little before it reaches the top is,

$$\vec{r}_i = -r\sin\tfrac{\Delta\theta}{2}\hat{i} + r\cos\tfrac{\Delta\theta}{2}\hat{j},$$

and the position of the tip of the second hand a little after it reaches the top is,

$$\vec{r}_f = r\sin\tfrac{\Delta\theta}{2}\hat{i} + r\cos\tfrac{\Delta\theta}{2}\hat{j}$$

where r is the length of the second hand. Using the definition of displacement,

$$\Delta\vec{r} \equiv \vec{r}_f - \vec{r}_i = \left(r\sin\tfrac{\Delta\theta}{2}\hat{i} + r\cos\tfrac{\Delta\theta}{2}\hat{j}\right) - \left(-r\sin\tfrac{\Delta\theta}{2}\hat{i} + r\cos\tfrac{\Delta\theta}{2}\hat{j}\right).$$

The y-components cancel leaving only the x-component, $\Delta\vec{r} = 2r\sin\tfrac{\Delta\theta}{2}\hat{i}$.

Using the small angle approximation, $\sin\tfrac{\Delta\theta}{2} \to \tfrac{\Delta\theta}{2} \Rightarrow \Delta\vec{r} = 2r\tfrac{\Delta\theta}{2}\hat{i} = r\Delta\theta\hat{i}$

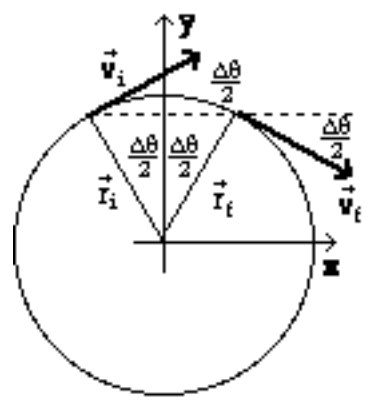
Using the definition of velocity, $\vec{v} \equiv \dfrac{\Delta\vec{r}}{\Delta t} = r\dfrac{\Delta\theta}{\Delta t}\hat{i}$.

In the limit as $\Delta t \to 0$, the instantaneous velocity becomes $\vec{v} = r\dfrac{d\theta}{dt}\hat{i}$.

Since the rate the angle is swept out is constant, $\vec{v} = r\dfrac{2\pi}{T}\hat{i} = \dfrac{2\pi r}{T}\hat{i}$.

Plugging in the numbers, $\vec{v} = \dfrac{2\pi(0.10)}{60}\hat{i} \Rightarrow \boxed{\vec{v} = (10.5\text{mm/s})\hat{i}}$.

Example 6.5: Find the acceleration of the tip of the second hand.



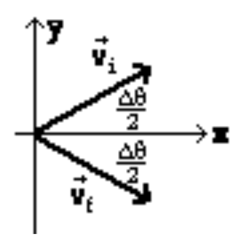
Given: v = 0.0105m/s and T = 60.0s.
Find: $\vec{v} = ?$

The velocity vectors are tangent to the circle at any point. Next, move the velocity just before and just after the top into standard position (lower sketch), so that their components are easier to find.

$$\vec{v}_i = v\cos\tfrac{\Delta\theta}{2}\hat{i} + v\sin\tfrac{\Delta\theta}{2}\hat{j} \text{ and } \vec{v}_f = v\cos\tfrac{\Delta\theta}{2}\hat{i} - v\sin\tfrac{\Delta\theta}{2}\hat{j}$$

where v is the speed of the tip of the second hand. The change in velocity is,

$$\Delta\vec{v} \equiv \vec{v}_f - \vec{v}_i = \left(v\cos\tfrac{\Delta\theta}{2}\hat{i} - v\sin\tfrac{\Delta\theta}{2}\hat{j}\right) - \left(v\cos\tfrac{\Delta\theta}{2}\hat{i} + v\sin\tfrac{\Delta\theta}{2}\hat{j}\right).$$



The x-components cancel leaving only the y-component,

$$\Delta\vec{v} = -2v\sin\tfrac{\Delta\theta}{2}\hat{j}.$$

Using the small angle approximation again,

$$\sin\tfrac{\Delta\theta}{2} \to \tfrac{\Delta\theta}{2} \Rightarrow \Delta\vec{v} = -2v\tfrac{\Delta\theta}{2}\hat{j} = -v\Delta\theta\hat{j}.$$

Substituting into the definition of acceleration, $\vec{a} \equiv \dfrac{\Delta\vec{v}}{\Delta t} = -v\dfrac{\Delta\theta}{\Delta t}\hat{j}$.

In the limit as $\Delta t \to 0$, the instantaneous acceleration becomes

$$\vec{a} = -v\frac{d\theta}{dt}\hat{j} = (-10.5)\frac{2\pi}{60}\hat{j} \Rightarrow \boxed{\vec{a} = 1.10\,mm/s^2(-\hat{j})}.$$

Notice that even though the speed of the second hand is constant, there is still an acceleration. Further, this acceleration is along the negative y-axis or toward the center. We'll into this more thoroughly in the section on uniform circular motion.

Section 6 - Summary

We can describe motion in two or three dimension with the exact same ideas we used for one dimension. The price we pay for the simplicity of using the same concepts of position, displacement, velocity, and acceleration is mathematical. These four quantities all become vectors.

Quantity	Definition	Mathematical Representation
Position	The location of an object with respect to a coordinate system.	$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$
Displacement	A change in position.	$d\vec{r} \equiv \vec{r}_f - \vec{r}_i = dx\hat{i} + dy\hat{j} + dz\hat{k}$
Velocity	The average rate of displacement.	$\vec{v} \equiv \frac{d\vec{r}}{dt} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$
Speed	The magnitude of the velocity.	$v \equiv \lvert\vec{v}\rvert = \sqrt{v_x^2 + v_y^2 + v_z^2}$
Acceleration	The rate of change of velocity.	$\vec{a} \equiv \frac{d\vec{v}}{dt} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$

In physics we try to explain everything with a few concepts as possible and let mathematics take care of the ever increasing complexity.