Section 9 - Uniform Circular Motion

Section Outline

1. The Speed and Acceleration
2. The Equations of Motion
3. A Summary of the Course to Date

What do objects do? They move. In this section we'll look at uniform circular motion such as the orbit of the moon. We really only have the definitions of position, displacement, velocity, and acceleration to guide us. However, that is enough to describe all motion, even motion in circles.

1. The Speed and Acceleration

Circular motion means constant radius, $\frac{dr}{dt} = 0$. That is, the position vector has a constant length. Uniform means constant speed, $\frac{dv}{dt} = 0$.

Let's think a bit more about this constant speed. The definition of speed is the distance covered per time. In this case the distance is around the circle, so it is just the circumference, $2\pi r$. The time to go around is called the period usually represented by T. So,

$$v \equiv \frac{\Delta s}{\Delta t} \Rightarrow v = \frac{2\pi r}{T}.$$

Since the direction of the velocity is tangent to the circle, this is called the "tangential speed."

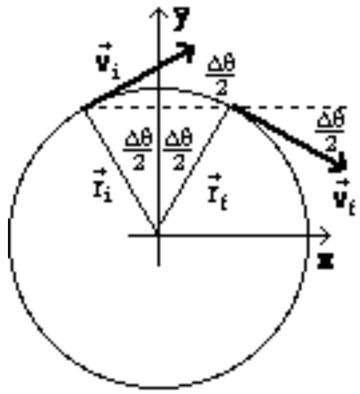
$$\boxed{\text{Tangential Speed} \quad v_t = \frac{2\pi r}{T}}$$

There are two other ideas associated with circular motion that we should mention just for completeness. Frequency is the number of orbits per unit time, so it is the reciprocal of the period,

$$f = \frac{1}{T}.$$

Angular frequency is the angle covered per unit of time. Angular frequency is often called "angular speed" just to add to the confusion. Since it is angle per time, it must the 2π divided by the period,

$$\omega \equiv \frac{\Delta\theta}{\Delta t} \Rightarrow \omega = \frac{2\pi}{T} = 2\pi f.$$

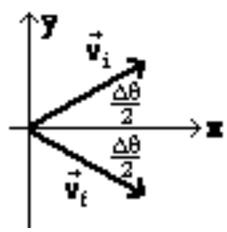


With those definitions established, let's move on and look back at our earlier study of the acceleration of the tip of the second hand on a clock.

For the second hand, the velocity vectors are tangent to the circle at any point. The velocity just before the top and just after the top are moved into standard position in the lower sketch, so that their components are easier to find.

$$\vec{v}_i = v\cos\tfrac{\Delta\theta}{2}\,\hat{i} + v\sin\tfrac{\Delta\theta}{2}\,\hat{j} \text{ and } \vec{v}_f = v\cos\tfrac{\Delta\theta}{2}\,\hat{i} - v\sin\tfrac{\Delta\theta}{2}\,\hat{j}$$

where v is the speed of the tip of the second hand.



The change in velocity is,

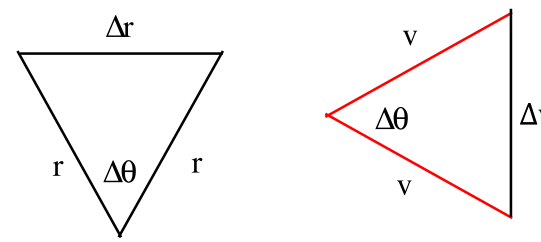
$$\Delta\vec{v} \equiv \vec{v}_f - \vec{v}_i = \left(v\cos\tfrac{\Delta\theta}{2}\,\hat{i} - v\sin\tfrac{\Delta\theta}{2}\,\hat{j}\right) - \left(v\cos\tfrac{\Delta\theta}{2}\,\hat{i} + v\sin\tfrac{\Delta\theta}{2}\,\hat{j}\right)$$

$$\Delta\vec{v} = -2v\sin\tfrac{\Delta\theta}{2}\,\hat{j} \Rightarrow d\vec{v} = -v\,d\theta\,\hat{j}\,.$$

Using the definition of acceleration,

$$\vec{a} \equiv \frac{d\vec{v}}{dt} = -v\frac{d\theta}{dt}\hat{j} = -\frac{v}{r}\frac{r\,d\theta}{dt}\hat{j} = -\frac{v}{r}\frac{ds}{dt}\hat{j} = -\frac{v^2}{r}\hat{j}\,.$$

That was a rather formal way to get the acceleration. A simpler method depends upon seeing the relationship between the two triangles shown above redrawn at the right. The first triangle is formed from the two position vectors for the second hand and the displacement vector. The second triangle is the two velocity vectors and the change in the velocity. Since both triangles are isosceles and have the same angle between the equal sides, they are similar. So,



$$\frac{\Delta v}{v} = \frac{\Delta r}{r} \Rightarrow \Delta v = \frac{v}{r}\Delta r\,.$$

Dividing both sides by the time to change the position (or velocity),

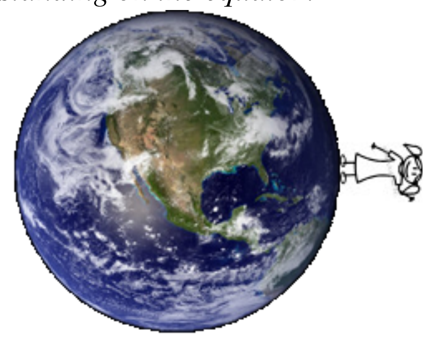
$$\frac{\Delta v}{\Delta t} = \frac{v}{r}\frac{\Delta r}{\Delta t} \Rightarrow a = \frac{v}{r}v \Rightarrow a = \frac{v^2}{r}\,.$$

In summary, the speed is constant, but there is acceleration toward the center called the "centripetal acceleration." The magnitude of the centripetal acceleration is,

$$\boxed{\text{Centripetal Acceleration} \quad a_c = \frac{v^2}{r}}$$

For the motion to remain uniform and circular there must be acceleration toward the center of precisely this magnitude.

Example 9.1: Find the (a)speed and (b)acceleration of a person standing on the equator.



Given: $R_E = 6.37x10^6$m and T = 1day = $8.64x10^4$s
Find: v = ? and a = ?

(a)The definition of speed is the distance traveled per time,

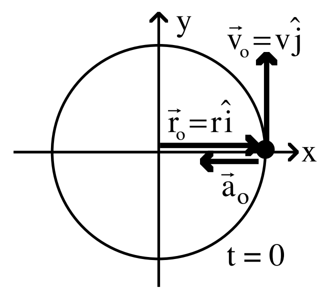
$$v \equiv \frac{\Delta x}{\Delta t} = \frac{2\pi R_E}{T} = \frac{2\pi(6.37x10^6)}{(24)(3600)} \Rightarrow \boxed{v = 463\,\tfrac{m}{s}}.$$

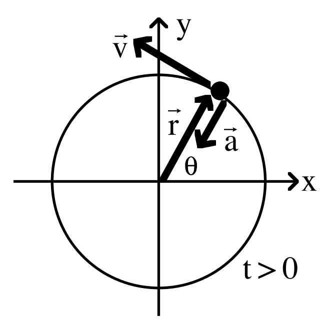
(b)The acceleration is centripetal,

$$a_c = \frac{v^2}{r} \Rightarrow a = \frac{v^2}{R_E} = \frac{(463)^2}{6.37x10^6} \Rightarrow \boxed{a = 0.0337\,\tfrac{m}{s^2}}.$$

2. The Equations of Motion

Now, let's go on to find the equations of motion. These are the equations that describe the acceleration, velocity and position as a function of time and the acceleration and velocity as a function of position for an object in uniform circular motion. In the same sense that the kinematic equations are the equations of motion for constant acceleration, these equations are the equations of motion for uniform circular motion. We will assume that initially the object is on the positive x-axis and its initial velocity is along the positive y-axis as shown at the right.





In the upper sketch at the right is an object in circular motion. The position, velocity, and acceleration vectors are shown. The lower sketch shows the object at some later time when the position vector makes an angle, θ, with the x-axis.

Since the motion is uniform,

$$\frac{d\theta}{dt} = \frac{2\pi}{T} = 2\pi f \equiv \omega$$

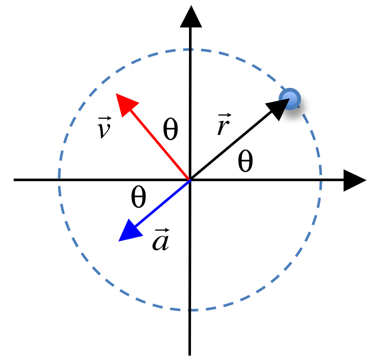
where T is the time to go around or the "period," f is the reciprocal of the period called the "frequency." The rate at which the angle is swept out is called the "angular frequency" and it is represented by the Greek letter ω (lower case omega).

The angle as a function of time can be found,

$$d\theta = \omega dt \Rightarrow \int_0^\theta d\theta = \int_0^t \omega dt = \omega \int_0^t dt \Rightarrow \theta = \omega t.$$

This is the easiest way to keep track of the angle as it changes with time. Rewriting the magnitudes of the velocity and acceleration terms of the angular speed,

$$v \equiv \frac{ds}{dt} = \frac{rd\theta}{dt} = r\omega \text{ and } a_c = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \omega^2 r.$$



The position, velocity, and acceleration vectors are redrawn in standard position at the right. The acceleration vector as a function of time can be written as,

$$\vec{a} = -a\cos\omega t\hat{i} - a\sin\omega t\hat{j} = -\omega^2 r\cos\omega t\hat{i} - \omega^2 r\sin\omega t\hat{j}.$$

Let's work our way through the x-components first,

$$a_x = -\omega^2 r\cos\omega t.$$

Note that the x-component is negative consistent with the sketch at the right. To find the x-component of the velocity, use the definition of acceleration,

$$a_x \equiv \frac{dv_x}{dt} \Rightarrow \frac{dv_x}{dt} = -\omega^2 r\cos\omega t \Rightarrow dv_x = -\omega^2 r\cos\omega t dt.$$

Now integrate keeping in mind the ω and r are constant,

$$\int_0^{v_x} dv_x = -\omega^2 r\int_0^t \cos\omega t dt \Rightarrow v_x = -\omega^2 r\left(\frac{1}{\omega}\sin\omega t\right) \Rightarrow v_x = -\omega r\sin\omega t.$$

Again, it matches the sketch. To find the x-component of the position, use the definition of velocity,

$$v_x \equiv \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -\omega r\sin\omega t \Rightarrow dx = -\omega r\sin\omega t dt.$$

Now integrate being careful with the limits,

$$\int_r^x dx = -\omega r\int_0^t \sin\omega t dt \Rightarrow x - r = -\omega r\left(-\frac{1}{\omega}[\cos\omega t - 1]\right) \Rightarrow x - r = r[\cos\omega t - 1] \Rightarrow x = r\cos\omega t.$$

To find the acceleration as a function of position, notice that,

$$a_x = -\omega^2 r\cos\omega t \Rightarrow a_x = -\omega^2 x.$$

To get the velocity as a function of position, square the velocity as a function of time,

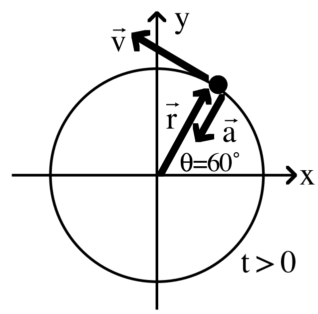
$$v_x^2 = \omega^2 r^2\sin^2\omega t = \omega^2 r^2\left(1-\cos^2\omega t\right) = \omega^2\left(r^2 - r^2\cos^2\omega t\right) = \omega^2\left(r^2 - x^2\right) \Rightarrow v_x = \pm\omega\sqrt{r^2 - x^2}$$

where the sign is chosen by the coordinates.

The equations for the y-components are derived in a similar manner. In summary,

$$\begin{aligned}
a_x(t) &= -\omega^2 r\cos\omega t & a_y(t) &= -\omega^2 r\sin\omega t \\
v_x(t) &= -\omega r\sin\omega t & v_y(t) &= \omega r\cos\omega t \\
x(t) &= r\cos\omega t & y(t) &= r\sin\omega t \\
a_x(x) &= -\omega^2 x & a_y(y) &= -\omega^2 y \\
v_x(x) &= \pm\omega\sqrt{r^2 - x^2} & v_y(y) &= \pm\omega\sqrt{r^2 - y^2}
\end{aligned}$$

Example 9.2: Using the coordinates at the right where the moon crosses the x-axis at t = 0. (a)Find the angular frequency. When the moon is at the position shown, find (b)the velocity, (c)the acceleration, and (d)the time. The moon's radius of orbit is $3.82x10^8$m and its period is 27.3 days.



Given: r = $3.82x10^8$m, θ = 60°, and T = 27.3days
Find: ω = ?, v = ?, a = ? and t = ?

(a)Using the angular speed,
$$\omega = \frac{2\pi}{T} = \frac{2\pi}{(27.3)(24)(3600)} \Rightarrow \boxed{\omega = 2.66 x 10^{-6} \text{rad/s}}.$$
The components of the position vector can be found from the equations of motion,
$$x = r\cos\omega t = (3.82x10^8)\cos 60° = 1.91x10^8 \text{m} \text{ and}$$
$$y = r\sin\omega t = (3.82x10^8)\sin 60° = 3.31x10^8 \text{m}.$$
(b)The components of the velocity at this position can be found using the equations of motion,
$$v_x(x) = \pm\omega\sqrt{r^2 - x^2} = -(2.66x10^{-6})\sqrt{(3.82x10^8)^2 - (1.91x10^8)^2} \Rightarrow v_x = -880 \text{m/s} \text{ and}$$
$$v_y(y) = \pm\omega\sqrt{r^2 - y^2} = +(2.66x10^{-6})\sqrt{(3.82x10^8)^2 - (3.31x10^8)^2} \Rightarrow v_y = 507 \text{m/s}.$$
The velocity is, $\boxed{\vec{v} = -(880\text{m/s})\hat{i} + (507\text{m/s})\hat{j}}$.
(c)The components of the acceleration at this position are again from the equations of motion,
$$a_x(x) = -\omega^2 x = -(2.66x10^{-6})^2(1.91x10^8) \Rightarrow a_x = -1.35 \text{mm/s}^s \text{ and}$$
$$a_y(y) = -\omega^2 y = -(2.66x10^{-6})^2(3.31x10^8) \Rightarrow a_y = -2.34 \text{mm/s}^s.$$
The acceleration is, $\boxed{\vec{a} = -(1.35\text{mm/s}^2)\hat{i} - (2.34\text{mm/s}^2)\hat{j}}$.
(d)Using the equation of motion for position as a function of time,
$$x(t) = r\cos\omega t \Rightarrow \omega t = \frac{\pi}{3} \Rightarrow t = \frac{\pi}{3\omega} = \frac{\pi}{3(2.66x10^{-6})} \Rightarrow \boxed{t = 3.94x10^5 \text{s} = 4.56 \text{days}}.$$

To summarize, an object in uniform circular motion has a tangential speed given by,
$$v_t = \frac{2\pi r}{T}.$$
The object is accelerating even though it is moving at a constant speed. The acceleration is due to the changing velocity vector. The magnitude of the acceleration required to maintain the uniform circular motion is given by,
$$a_c = \frac{v^2}{r}.$$

The equations of motion for uniform circular motion are:

$$a_x(t) = -\omega^2 r\cos\omega t \qquad a_y(t) = -\omega^2 r\sin\omega t$$
$$v_x(t) = -\omega r\sin\omega t \qquad v_y(t) = \omega r\cos\omega t$$
$$x(t) = r\cos\omega t \qquad y(t) = r\sin\omega t$$
$$a_x(x) = -\omega^2 x \qquad a_y(y) = -\omega^2 y$$
$$v_x(x) = \pm\omega\sqrt{r^2 - x^2} \qquad v_y(y) = \pm\omega\sqrt{r^2 - y^2}$$

3. A Summary of the Course to Date

We can now describe what objects do. That is, we can describe their motion. Now, we can begin to fill in the outline for the course.

- Classical Mechanics
  - Translation — The motion of objects as a whole
    - Kinematics
      - Laws: none
      - Definitions:
        - Position $\vec{r}$
        - Displacement $d\vec{r}$
        - Velocity $\vec{v} \equiv \frac{d\vec{r}}{dt}$
        - Acceleration $\vec{a} \equiv \frac{d\vec{v}}{dt}$
      - Useful Relations:
        - Kinematic Equations
          - $x = x_o + v_o t + \frac{1}{2}at^2$
          - $v = v_o + at$
          - $v^2 = v_o^2 + 2a(x - x_o)$
          - $x - x_o = \frac{1}{2}(v + v_o)t$
        - Tangential Speed $v_t = \frac{2\pi r}{T}$
        - Centripetal Acceleration $a_c = \frac{v^2}{r}$
    - Force
    - Momentum
    - Energy
  - Rotation — The spinning motion of objects
    - Kinematics
    - Torque
    - Angular Momentum
    - Energy

Now its time to figure out the cause of motion!