Section 14 – Forces in Circular Motion

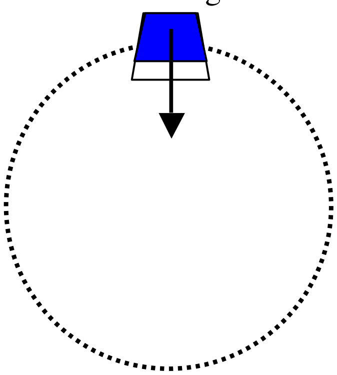
Outline

1. Uniform Circular Motion
2. Non-uniform Circular Motion

Why do objects do what they do? The answer we have been investigating is forces. If forces can explain motion, then they must be able to explain circular motion.

1. Uniform Circular Motion

You have probably never tried to swing a bucket of water over your head, but it can be done quite easily (http://www.youtube.com/watch?v=bWiVv--OTYA). Why doesn't the water fall out on your head? For that matter, why doesn't the orbiting moon fall toward Earth? Why do we keep orbiting the sun and not fall inward and get burned to a crisp? The answer must be contained in Newton's Laws of Motion.



According to the First Law, the water will move in a straight line unless a force acts on it. Therefore, there must always be a force on an object to keep it in circular motion.

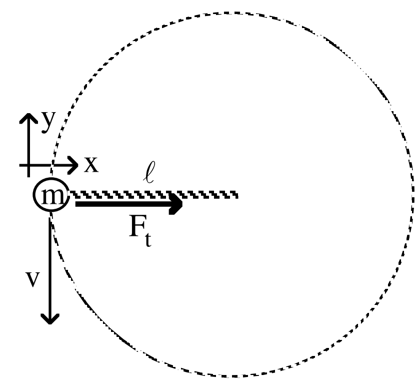
According to the Second Law, the force and the acceleration must point in the same direction. For objects in uniform circular motion (constant speed) we already learned that there is acceleration toward the center called the centripetal acceleration,

$$a_c = \frac{v_t^2}{r}.$$

Therefore, there must be a force acting toward the center to keep an object in circular motion. In the case of the water, it is the force of gravity combined with the normal force from the bucket. In the case of the moon orbiting Earth, it is the force of gravitation. The point is, objects in circular motion can be treated the same way we've treated other objects. Here are some examples.

Example 14.1: A 100g ball is twirled overhead on the end of a 40.0cm string at 100rpm. Find the tension in the cord.

Given: $m = 0.100\text{kg}$, $\ell = 0.400\text{m}$, and $f = 100\text{rpm}$
Find: $F_t = ?$



The only force on the ball is the tension. Choosing the x-axis as shown and applying Newton's Second Law,

$$\Sigma F_x = ma_x \Rightarrow F_t = ma_c .$$

Using the centripetal acceleration, $F_t = m\frac{v_t^2}{r}$.

Using the tangential speed,

$$v_t = \frac{2\pi r}{T} \Rightarrow F_t = m\frac{1}{r}\left(\frac{2\pi r}{T}\right)^2 \Rightarrow F_t = m\frac{4\pi^2 r}{T^2}.$$

The period is the reciprocal of the frequency,

$$T = \frac{1}{f} = \frac{1}{100\frac{\text{rev}}{\text{min}}} = \frac{1\text{min}}{100\text{rev}} = \frac{60.0\text{s}}{100\text{rev}} = 0.600\text{s},$$

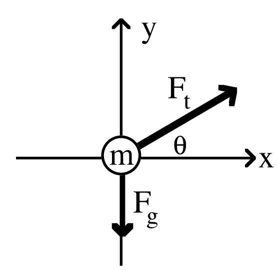
and the radius is the length of the string. Finally,

$$F_t = m\frac{4\pi^2\ell}{T^2} = (0.100)\frac{4\pi^2(0.400)}{(0.600)^2} \Rightarrow \boxed{F_t = 4.39\text{N}}.$$

But wait (weight?) we forgot about gravity! The side view of the twirling ball is shown at the right. Notice that the ball is moving in a plane slightly below the point where the string is held. In this more careful analysis, there are forces along two directions and only the horizontal part of the tension causes the circular motion. Applying the Second Law to each direction separately,

$$\Sigma F_x = ma_x \Rightarrow F_t\cos\theta = m\frac{v_t^2}{r} \Rightarrow F_t = m\frac{v_t^2}{r\cos\theta} \quad (1).$$

$$\Sigma F_y = ma_y \Rightarrow F_t\sin\theta - F_g = 0 \Rightarrow F_t\sin\theta = F_g \quad (2).$$

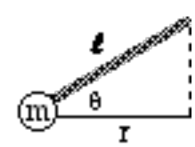


Substituting for the speed in eq. 1,

$$F_t = m\frac{1}{r\cos\theta}\left(\frac{2\pi r}{T}\right)^2 = \frac{4\pi^2 mr}{T^2\cos\theta}.$$

Notice that from trigonometry,

$$r = \ell\cos\theta \Rightarrow F_t = \frac{4\pi^2 m\ell\cos\theta}{T^2\cos\theta} = \frac{4\pi^2 m\ell}{T^2} \Rightarrow \boxed{F_t = 4.39\text{N}}.$$



The tension is the same as before! Since the component of the tension toward the center of the circle is less, the ball must be moving slower. In fact, the speed depends on the angle below horizontal,

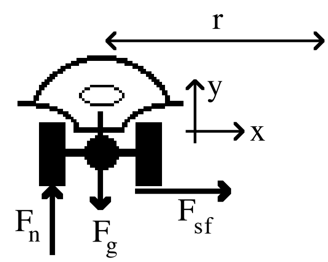
$$v_t = \frac{2\pi r}{T} = \frac{2\pi\ell\cos\theta}{T}.$$

The faster it spins the smaller the angle, as you might expect.
The angle below the horizontal can be found from eq. 2,

$$\sin\theta = \frac{F_g}{F_t} = \frac{mg}{F_t} \Rightarrow \theta = \arcsin\left(\frac{mg}{F_t}\right) = \arcsin\left[\frac{(0.100)(9.80)}{4.39}\right] \Rightarrow \boxed{\theta = 12.9^\circ}.$$

Example 14.2: A car traveling at 50.0km/h rounds a curve with a 30.0m radius. Find the minimum coefficient of friction required to keep the car from skidding.

Given: v = 50.0km/h = 13.9m/s and r = 30.0m
Find: μ = ?



The forces on the car are shown at the right. They are the weight, normal and static friction. The friction is the force that causes the car to go in a circle and points toward the center. You know this because turning corners on an icy road is dangerous. Applying the Second Law to each direction separately,

$$\Sigma F_x = ma_x \Rightarrow F_{sf} = ma_c \quad (1).$$

$$\Sigma F_y = ma_y \Rightarrow F_n - F_g = 0 \Rightarrow F_n = F_g \quad (2).$$

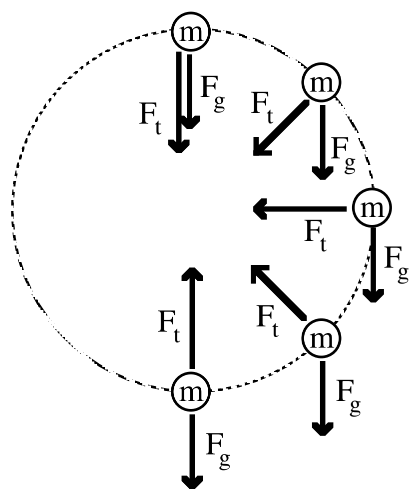
Using the definition of the coefficient of static friction,

$$\mu_s \equiv \frac{F_{sf,max}}{F_n} \Rightarrow F_{sf} \leq \mu_s F_n \Rightarrow ma_c \leq \mu_s F_g \Rightarrow \mu_s \geq \frac{ma_c}{F_g}.$$

Applying the mass/weight rule and the expression for centripetal acceleration,

$$\mu_s \geq \frac{\not{m}a_c}{\not{m}g} = \frac{v^2}{rg} = \frac{(13.9)^2}{(30.0)(9.80)} \Rightarrow \boxed{\mu_s \geq 0.657}.$$

2. Non-uniform Circular Motion

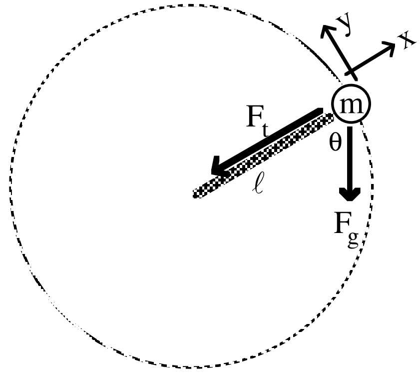


If you twirl a ball at the end of a string slowly in a vertical plane, instead of a horizontal plane, you'll notice the speed isn't constant. The ball will be going slower at the top and faster at the bottom.

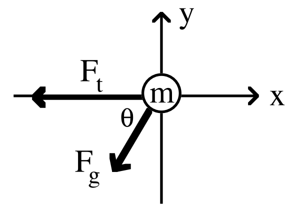
Circular motion doesn't always occur with constant speed. Non-uniform circular motion occurs in many cases. Since the speed changes, the acceleration must have a component along the motion as well as the usual component toward the center. The component along the motion is called the "tangential acceleration."

This too can be understood using Newton's Laws. While the tension force always points toward the center, the weight has tangential components.

Example 14.3: When the ball is 60.0° from the vertical moving downward, the tension is 5.00N. Find (a)the centripetal acceleration, (b)the tangential acceleration and (c)the tangential velocity.



Free Body Diagram



Given: $\theta = 60.0°$ and $F_t = 5.00N$
Find: $a_c = ?$, $a_t = ?$, and $v_t = ?$

Using the free body diagram to apply the Second Law,

$$\Sigma F_x = ma_x \Rightarrow -F_t - F_g\cos\theta = ma_c \quad (1).$$

$$\Sigma F_y = ma_y \Rightarrow -F_g\sin\theta = ma_t \quad (2).$$

(a)Using the mass/weight rule in eq. 1 and solving for the centripetal acceleration,

$$-F_t - mg\cos\theta = ma_c \Rightarrow a_c = -\left(\frac{F_t}{m} + g\cos\theta\right).$$

Putting in the numbers,

$$a_c = -\left[\frac{5.00}{0.100} + (9.80)\cos 60.0°\right] \Rightarrow \boxed{a_c = -54.9\,\mathrm{m/s^2}}.$$

The minus sign is because the x-axis points away from the center of the circle.

(b)The tangential acceleration can be found using eq. 2,

$$-\not{m}g\sin\theta = \not{m}a_t \Rightarrow a_t = -g\sin\theta = (-9.80)\sin 60.0° \Rightarrow$$

$$\boxed{a_t = -8.49\,\mathrm{m/s^2}}.$$

Again the minus sign is due to the choice of axes.

(c)The tangential velocity is the velocity that is related to the centripetal acceleration,

$$a_c = \frac{v_t^2}{r} \Rightarrow v_t = \sqrt{a_c r} = \sqrt{(54.9)(0.400)} \Rightarrow \boxed{v_t = 4.69\,\mathrm{m/s}}.$$

In summary, Newton's Laws of Motion show us that circular motion requires a force toward the center to create the centripetal acceleration. Tangential forces change the speed of the circular motion.

Now, back to the question about the moon staying in orbit. Earth exerts a gravitational force on the moon. This force pulls it toward Earth. If the moon were not moving, the gravitational force would cause the moon to fall directly toward Earth. However, the moon is not at rest, it moving tangentially. Think about a baseball thrown horizontally, it falls toward Earth, but doesn't land directly below the point from which it is thrown because of its horizontal motion. If you could throw the ball faster, it will travel farther before hitting the ground. If you could throw it fast enough, it would fall toward Earth at exactly the same rate the spherical shape of Earth causes the surface to fall away from the ball. The ball would be in orbit! So the moon is moving at just the right speed so that as it "falls toward Earth" but it never gets any closer. This is why there has to be a very special relationship between the velocity, radius, and acceleration for an object in uniform circular motion. You know this relationship,

$$a_c = \frac{v_t^2}{r}.$$

You might ask how it has come to pass that the moon has this exact right velocity. For that matter, why does every planet in the solar system have the perfect velocity to maintain its orbit around the sun? Remember, there are some objects in the solar system that don't have circular orbits, such as comets. In 1994 a comet collided with Jupiter (http://www2.jpl.nasa.gov/sl9/). This is the eventual fate of all objects that don't have circular orbits. The reason that we only really see planets and moons with circular orbits, is that the solar system is old enough that most of the things with non-circular orbits have smashed into something already.

Speaking of falling toward earth without getting any closer, this is why astronauts feel weightless. The force of gravity on Space Shuttle astronauts is only about 10% less than the force of gravity on Earth. They are falling toward Earth all the time, but so is the shuttle. That is why they can float around inside - everything is falling together. Do an internet search for "vomit comet." You'll find information about an airplane they use to train astronauts (and make movies). The plane goes up to high altitude and then "falls" in a parabolic trajectory. Since the passengers are falling in the same way the plane is, they feel "weightless."

Section Summary

Our goal is to understand what objects do and why they do it. Now we have applied Newton's Laws to understand the causes of circular motion. A net force toward the center of the circle is required for circular motion. Any forces tangential to the circle cause changes in the orbital speed.