**Section 17 – Center of Mass**

Section Outline

1. The Second Law and the Center of Mass
2. Calculating the Center of Mass for Extended Objects

We are trying to understand why objects do what they do. Up until now, we have only used the concept of force to answer the question. Now, we are trying to add the idea of momentum to the mix so that we can deal with systems that contain many objects. This brings up several questions regarding the use of the Second Law for systems composed of many objects. When writing the Second Law as \( F = ma \):

1. Which forces count and which don’t?
2. What mass do you use, the total mass, the average mass, or what?
3. What acceleration are we talking about, to which object does it apply?

The answers are:

1. Forces that act on the system from the outside, not forces that act between objects within the system.
2. The mass is the total mass of all the objects in the system.
3. The acceleration is the acceleration of the center of mass of the system.

Let’s build some understanding of these answers and learn to calculate the location of the center of mass.

**1. The Second Law and the Center of Mass**

When we write down the Second Law for a system that contains a single object such as a tossed ball it is pretty clear what we mean. The sum of the forces only includes forces acting on the ball, in this case gravity. The mass is the mass of the ball and the acceleration in the rate of change of velocity of the ball, which turns out to equal \( g \).

\[
\sum F = m\ddot{a} \Rightarrow \sum F_g = m\ddot{a} \Rightarrow -mg\hat{j} = m\ddot{a} \Rightarrow \ddot{a} = -g\hat{j}
\]

However, for a system that contains more than one object, the answer is less clear. Consider a baton, a stick with a mass on each end as shown at the right. There are lots of forces; the weight of each ball, the weight of the stick, the force that each ball exerts on the stick the forces that the stick exerts on each ball. Which forces count? Which mass should be used, the mass of one ball, the mass of both, the mass of everything? Does the acceleration refer to the motion of the first ball, the second ball, or to the system as a whole somehow? In short, just what does the Second Law mean for a system of objects?

Let’s look at the question of which forces count. In one sense, all the forces count, but the net result is that the only forces that count are applied externally on the system such as the weight. Internal forces, such as the force exerted by the stick on the first mass, are canceled by the force that the mass exerts back on the stick. By the Third Law, all such pairs of internal action/reaction forces always cancel, leaving only forces that act from the outside.
To understand what mass counts, look at the original Second Law,
\[ \Sigma \vec{F} = \frac{d\vec{p}}{dt} . \]
We can find the momentum of the system by adding up the momentum of each part,
\[ \vec{p} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3. \]
This can be set equal to the total mass of the system times some velocity,
\[ m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 = \frac{m_1 \vec{v}_1 + m_2 \vec{v}_2 + m_3 \vec{v}_3}{M} \]
This velocity is a type of average called a “weighted” average (ironic since it is really a “massed” average.

**A Mathematical Moment - Weighted Average**

Suppose on a test two students got 60, five earned 70, and three had 80. You might find the average by doing the calculation,
\[ \frac{60 + 60 + 70 + 70 + 70 + 70 + 80 + 80 + 80}{10} = 7.1 \]
You could also get the answer by using,
\[ \frac{2(60) + 5(70) + 3(80)}{10} = 7.1. \]
If you “weigh” each score by the number getting that score, you have performed a “weighted average.”

The velocity you get when you calculate the weighted average velocity is the “center-of-mass velocity.” You could differentiate this result if you want to find the center-of-mass acceleration. Instead, let’s integrate the center-of-mass velocity to get the position of the center-of-mass, the result is,
\[ \vec{r}_{cm} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + m_3\vec{r}_3}{M}, \]
for three masses. For any number of masses, N,

**The Definition of the Center-of-Mass**
\[ \vec{r}_{cm} \equiv \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i. \]

Since this is a vector equation, it can be broken down separately for each axis,
\[ x_{cm} = \frac{1}{M} \sum_{i=1}^{N} m_i x_i \text{, } y_{cm} = \frac{1}{M} \sum_{i=1}^{N} m_i y_i \text{, and } z_{cm} = \frac{1}{M} \sum_{i=1}^{N} m_i z_i. \]

**Example 17.1:** Find the center of mass of the baton assuming the mass of stick can be ignored and the other two masses are equal.

Using the definition of center of mass,
\[ x_{cm} \equiv \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m(0) + m(\ell)}{m + m} = \frac{m(\ell)}{2m} \Rightarrow x_{cm} = \frac{\ell}{2} \]
as you might have guessed.
The point of all this is that when you write Newton’s Second Law for a system of objects you can use the \( F = ma \) form or the original form, whichever is easier understanding that:

- The only forces that count are ones that act on the system of objects from the outside.
- The mass is the mass of the entire system of objects.
- The acceleration is the acceleration of the center-of-mass of the system.

Therefore, the Second Law determines the motion of the center-of-mass of the system to which it is applied.

**Example 17.2:** Sketch the motion of the tossed baton with special attention to the center-of-mass.

Since the Second Law predicts the motion of the center-of-mass and we know the motion of an object that only feels gravity is a parabola, the center-of-mass of the baton will follow a parabolic path. The masses may rotate about that center-of-mass. So, as we work toward completing our understanding of motion we will have to eventually take rotation into account.

### 2. Calculating the Center of Mass for Extended Objects

The center-of-mass is not too difficult to find for a set of discreet masses. It is a bit more challenging for an extended object like a baseball bat. The center-of-mass of the bat could be found by breaking the bat up into a collection of small masses and summing them up as usual,

\[
\vec{r}_{cm} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i .
\]

The question is, how small do the masses need to be? The answer is, the smaller the better. If we let them get infinitesimally small, then the masses are written using the calculus as \( dm \) and the sum becomes an integral,

\[
\vec{r}_{cm} = \frac{1}{M} \int \vec{r} \, dm.
\]

Again, this can be written separately for each coordinate axis,

\[
x_{cm} = \frac{1}{M} \int x \, dm , \quad y_{cm} = \frac{1}{M} \int y \, dm , \quad \text{and} \quad z_{cm} = \frac{1}{M} \int z \, dm .
\]
For now, we will concentrate on just one axis.

**Example 17.3:** Find the center-of-mass for a bat modeled as a wedge 80.0 cm long, 2.00 cm wide on one side and 8.00 cm wide on the other.

Given: \( b = 1.00 \text{ cm}, \ a = 4.00 \text{ cm} \).

Find: \( x_{cm} = ? \)

![Diagram of a wedge](image)

The center-of-mass of the wedge is along the x-axis by the symmetry of the situation, so we really only need to find the x-coordinate of the center-of-mass, which is,

\[ x_{cm} = \frac{1}{M} \int x \, dm. \]

To do the integral we need to choose a \( dm \) that has a constant value of \( x \), so we will use a thin vertical slice. Next we need to express \( dm \) in terms of the thickness of the slice, \( dx \), so that the integral is all in terms of a single variable, \( x \). We can do this because the fraction of the mass in the wedge is the same as the fraction of the area in the wedge. Note that \( 2y \) is the height if the slice and the thickness is \( dx \), so

\[ dm = \frac{2y}{A} dx. \]

The area of the wedge is,

\[ A = (a + b) \ell \Rightarrow dm = \frac{2y}{M(a + b)\ell} dx. \]

Substituting into the definition of center-of-mass,

\[ x_{cm} = \frac{1}{M} \int x A \frac{2y}{(a + b)\ell} dx = \frac{2}{(a + b)\ell} \int xydx. \]

Next, we need to express \( y \), the height of the wedge, as a function of \( x \). This is a straight line given by the equation,

\[ y = \frac{a + b}{\ell} x + b \Rightarrow x_{cm} = \frac{2}{(a + b)\ell} \int (\frac{a + b}{\ell} x^2 + bx)dx. \]

Finally, we can put on the limits and integrate,

\[ x_{cm} = \int_0^\ell (\frac{a + b}{\ell} x^2 + bx)dx = \frac{2}{(a + b)\ell} \left[ \frac{a + b}{3\ell} x^3 + \frac{b}{2} x^2 \right]_0^\ell = \frac{2}{(a + b)\ell} \left( \frac{a + b}{3\ell} \ell^3 + \frac{b}{2} \ell^2 \right) \Rightarrow \]

\[ x_{cm} = \frac{\ell}{3} \left( \frac{2a + b}{a + b} \right). \]

Plugging in the numbers,

\[ x_{cm} = \frac{80}{3} \left( \frac{2(4) + 1}{4 + 1} \right) \Rightarrow x_{cm} = 48 \text{ cm}. \]

That was rather challenging! At least we got an answer that was more than halfway along the bat. Another way to check our answer is to use the method called “limiting cases.” If \( a = b \), then we would have a rectangle and the center-of-mass would be in the middle. Let’s check this,

\[ x_{cm} = \frac{\ell}{3} \left( \frac{2a + b}{a + b} \right) \Rightarrow x_{cm} = \frac{\ell}{3} \left( \frac{2b + b}{b + b} \right) = \frac{\ell}{3} \left( \frac{3b}{2b} \right) = \frac{\ell}{2}. \]
COMMENT ON PROBLEM SOLVING:
One great way to check your answer on a problem is to find and check “limiting cases.”

Section Summary
The Original Second Law helped us understand that the Second Law really refers to the center-of-mass of an object. That is, the sum of the forces that act on an object, or system of objects from the outside is equal to the total mass of the system times the acceleration of the center-of-mass.

The definition of the center of mass is,

\[ \mathbf{r}_{cm} \equiv \frac{1}{M} \int \mathbf{r} \, dm. \]