Section 18 – The Law of Conservation of Momentum

Section Outline

1. The Law of Conservation of Linear Momentum
2. Linear Momentum and Collisions

Linear Momentum turns out to be a very powerful idea for extending our understanding of why objects do what they do. In this section we'll see that under certain common conditions, the total momentum of a system of object remains constant regardless of the actions of the objects in the system. For example, we can understand the collision between two pool balls because their combined momentum remains constant.

1. The Law of Conservation of Linear Momentum

Consider a system of objects that, at most, exert forces on each other. In other words, there are no external forces on this system. Applying the Original Second Law to this system,

$$\Sigma \vec{F} = \frac{d\vec{p}}{dt} \Rightarrow 0 = \frac{d\vec{p}}{dt}$$

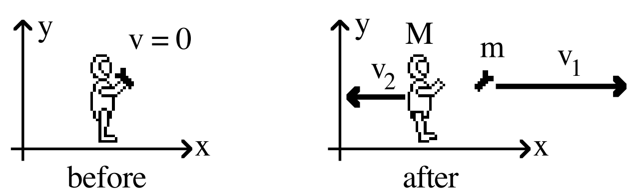
No external forces means that the total linear momentum of the system doesn't change with time. The linear momentum of the system must remain constant. This is the Law of Conservation of Linear Momentum.

The Law of Conservation of Linear Momentum
"The total linear momentum of an isolated systems of objects remains constant."

Example 18.1: A 100kg astronaut throws a 1.00kg wrench at 25.0m/s. Find the recoil velocity of the astronaut.

Given: m = 1.00kg, M = 100kg, and v_1 = 25.0m/s
Find: v_2 = ?

The system of the astronaut and the wrench is isolated in the sense that the only forces acting are the force the astronaut exerts on the wrench and the equal and opposite reaction force that the wrench exerts on the astronaut. So, the Law of Conservation of Momentum can be used.



The initial momentum of the system is zero, $p_o = 0$.
The final momentum of the system is, $p = mv_1 - Mv_2$. Notice that momentum is a vector quantity so motion toward the left is negative according to the coordinate system shown.
According to the Law of Conservation of Momentum, these can be set equal to each other.

$$p_o = p \Rightarrow 0 = mv_1 - Mv_2 \Rightarrow v_2 = \frac{m}{M} v_1 .$$

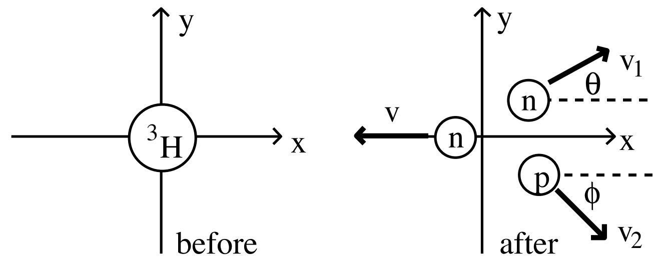
Putting the numbers in,

$$v_2 = \frac{1.00}{100}(25.0) \Rightarrow \boxed{v_2 = 0.250\,m/s}.$$

COMMENT ON PROBLEM SOLVING:
When a problem involves the Law of Conservation of Linear Momentum, the best way to approach it is with two pictures; one of the initial situation, the "before" picture, and one with the final state, the "after" picture. Then you can work on getting an expression for the momentum in each picture and set them equal.

Let's look deeper into the vector nature of linear momentum.

Example 18.2: A tritium nucleus (^3H) at rest in a fusion reactor breaks up into a neutron traveling to the left at 75.0km/s, a second neutron that heads off at a 30.0° angle with a speed of 40.0km/s, and a proton. Find the velocity of the proton assuming the masses of the neutron and proton are equal.



Given: $v = 75.0km/s$, $v_i = 40.0km/s$,
$\theta = 30.0°$, and $m_p = m_n = m$
Find: $v_2 = ?$ and $\phi = ?$

Momentum is a vector, so both the x and y components must be considered. The initial momentum of the system is zero along each axis,

$$p_{ox} = 0 \text{ and } p_{oy} = 0.$$

The final momentum is,

$$p_x = mv_1\cos\theta + mv_2\cos\phi - mv$$

and

$$p_y = mv_1\sin\theta - mv_2\sin\phi.$$

Applying the Law of Conservation of Momentum,

$$0 = mv_1\cos\theta + mv_2\cos\phi - mv \Rightarrow v_2\cos\phi = v - v_1\cos\theta$$
$$0 = mv_1\sin\theta - mv_2\sin\phi \Rightarrow v_2\sin\phi = v_1\sin\theta$$

where the mass has been cancelled out. Putting in the numbers,

$$v_2\cos\phi = 75.0 - (40.0)\cos 30.0° = 40.4km/s \quad (1)$$
$$v_2\sin\phi = (40.0)\sin 30.0° = 20.0km/s \quad (2)$$

These are the components of the velocity of the proton. The magnitude can be found using the Pythagorean Theorem,

$$v_2 = \sqrt{(v_2\cos\phi)^2 + (v_2\sin\phi)^2} = \sqrt{(40.4)^2 + (20.0)^2} \Rightarrow \boxed{v_2 = 45.1km/s}.$$

The angle can be found from the definition of tangent,

$$\phi = \arctan\left(\frac{v_2\sin\phi}{v_2\cos\phi}\right) = \arctan\left(\frac{20.0}{40.4}\right) \Rightarrow \boxed{\phi = 26.3°}.$$

The Law of Conservation of Momentum allowed us to figure out the results of this interaction without needing to know any details about the forces between the interacting parts of the system because the Third Law promises that they will cancel. So, even though we don't know anything about the strength or behavior of nuclear forces, we can make predictions about a nuclear interaction.

2. Linear Momentum and Collisions

The word "collision" implies that the time for the event is very short. Therefore, external forces have little time to act. In other words, the impulse to the system due to external forces can be assumed to be zero. Then, according to the Impulse-Momentum Theorem, the change in momentum must also be zero. That is, momentum is conserved.

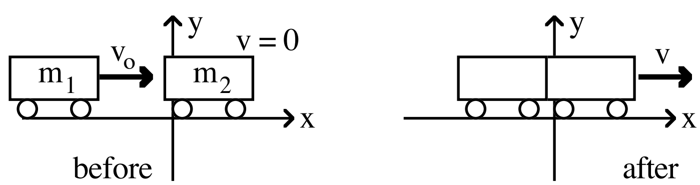
Think about this in terms of the baseball colliding with a bat. During the collision there are actually external forces acting on the ball and bat. The ball feels a gravitational force of about 1.5N, the bat feels a gravitational force of about 10N, and the batter is exerting a force on the bat of about 50N. Each of these forces changes the momentum of the ball-bat system. The largest impulse is due to the batter. Since the collision takes about 1.0ms, the impulse is,

$$J \equiv \int F dt \approx F\Delta t \approx (50)(10^{-3}) \approx 0.05 N \cdot s.$$

However, the change in momentum of a well hit ball is about 12N·s. So, the external forces acting on the system change the total momentum of the system by a small fraction of the momentum involved. This allows us to neglect the external forces and assume that momentum is conserved.

Example 18.3: A 2000kg railroad car is coupled to a second car of mass 1500kg by gently colliding with it at 2.50m/s. Find the speed of the cars just after collision.

Given: m_1 = 2000kg, m_2 = 1500kg, and v_o = 2.50m/s
Find: v = ?



The momentum before the collision is, $p_o = m_1 v_o$.
After collision, $p_o = (m_1 + m_2)v$.
Law of Conservation of Momentum requires,

$$m_1 v_o = (m_1 + m_2)v.$$

Solving for v,

$$v = \frac{m_1}{m_1 + m_2} v_o,$$

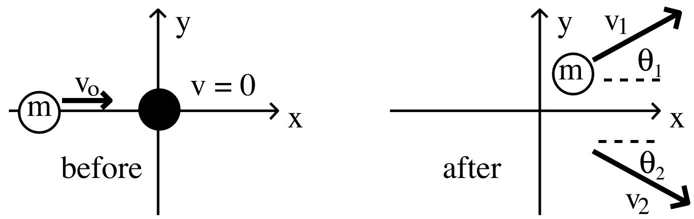
and putting in the numbers,

$$v = \frac{2000}{2000 + 1500}(2.50) \Rightarrow \boxed{v = 1.43 m/s}.$$

Now, let's look at a two dimensional collision.

Example 18.4: The cue ball is traveling at 4.80m/s when it collides with the eight ball. The cue heads off at 4.07m/s at 32.0° to its original direction. Find the velocity of the eight ball after the collision.

Given: v_o = 4.80m/s, v_1 = 4.07m/s, and θ_1 = 32.0°
Find: v_2 = ? and θ_2 = ?



$$p_{ox} = mv_o \qquad p_{oy} = 0$$

$$p_x = mv_1\cos\theta_1 + mv_2\cos\theta_2 \qquad p_y = mv_1\sin\theta_1 - mv_2\sin\theta_2$$

Recall that momentum is a vector quantity, so each component must be kept track of separately.

The Law of Conservation of Momentum applies to each direction independently,

$$mv_o = mv_1\cos\theta_1 + mv_2\cos\theta_2 \Rightarrow v_2\cos\theta_2 = v_o - v_1\cos\theta_1 \qquad (1)$$

$$0 = mv_1\sin\theta_1 + mv_2\sin\theta_2 \Rightarrow v_2\sin\theta_2 = v_1\sin\theta_1 \qquad (2)$$

Here is a bit of tricky algebra to solve for v_2 and θ_2. Dividing eq. 2 by eq. 1,

$$\frac{v_2\sin\theta_2}{v_2\cos\theta_2} = \frac{v_1\sin\theta_1}{v_o - v_1\cos\theta_1} \Rightarrow \tan\theta_2 = \frac{v_1\sin\theta_1}{v_o - v_1\cos\theta_1} \Rightarrow \theta_2 = \arctan\frac{v_1\sin\theta_1}{v_o - v_1\cos\theta_1}$$

Plugging in the numbers,

$$\theta_2 = \arctan\left(\frac{(4.07)\sin 32}{4.80 - (4.07)\cos 32}\right) \Rightarrow \boxed{\theta_2 = 58.0}.$$

Squaring eq. 2 and adding it to the square of eq. 1,

$$(v_2\sin\theta_2)^2 + (v_2\cos\theta_2)^2 = (v_1\sin\theta_1)^2 + (v_o - v_1\cos\theta_1)^2$$

$$v_2^2(\sin^2\theta_2 + \cos^2\theta_2) = v_o^2 + v_1^2(\sin^2\theta_1 + \cos^2\theta_1) - 2v_o v_1\cos\theta_1$$

$$\Rightarrow v_2^2 = v_o^2 + v_1^2 - 2v_o v_1\cos\theta_1$$

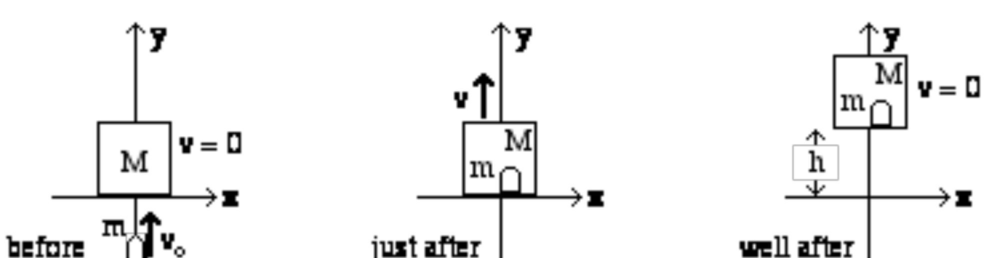
Plugging in the numbers,

$$v_2 = \sqrt{(4.80)^2 + (4.07)^2 - 2(4.80)(4.07)\cos 32.0} \Rightarrow \boxed{v_2 = 2.54\,\mathrm{m/s}}.$$

Notice that the balls diverge from each other at the 90° angle you might expect if you've played pool.

Example 18.5: A 60.0g bullet fired vertically into a 4.00kg block of wood at 400m/s sticks in the block. Find (a)the speed of the block just after the collision and (b)maximum height of the block.

Given: m = 0.0600kg, M = 4.00kg, and v_o = 400m/s
Find: h = ?



$p_o = mv_o$ $\qquad$ $p = (m + M)v$ $\qquad$ $p = 0$ Where does the momentum go?

(a)The Law of Conservation of Momentum can be applied just before until just after the collision because during this short time gravity provides a negligible impulse. Therefore,

$$mv_o = (m+M)v \Rightarrow v = \frac{m}{m+M}v_o$$

Putting in the numbers,

$$v = \frac{0.0600}{0.0600+4.00}(400) \Rightarrow \boxed{v = 5.91m/s}.$$

Now, the momentum begins to disappear because the force of gravity has time to act.

(b)This now becomes a freefall problem with the speed from part (a) becoming the initial speed and the final speed at the top being zero. We can get the height using the kinematic equation without the time,

$y_o = 0$
$y = ?$
$v_o = 5.91$m/s
$v = 0$
$a = -9.80$m/s^2
$t = ?$

$$v^2 = v_o^2 + 2a(y - y_o) \Rightarrow 0 = v_o^2 + 2ay \Rightarrow y = -\frac{v_o^2}{2a}$$

Putting in the numbers,

$$y = -\frac{v_o^2}{2a} = -\frac{(5.19)^2}{2(-9.80)} \Rightarrow \boxed{h = 1.78\text{m}}.$$

<u>Section Summary</u>

When a system of objects is "isolated" that is, it feels no forces from outside the system, the total momentum of the system stays fixed.

The Law of Conservation of Linear Momentum

"The total linear momentum of an isolated systems of objects remains constant."

The Law of Conservation of Linear Momentum is very useful for understanding collisions.

A Summary of the Course to Date

What do objects do? Why do they do it? We've now added momentum to the list of concepts that describe why objects do what they do.