Section 19 – The Definition of Work

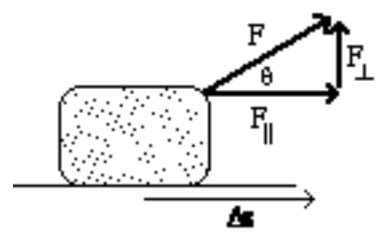
Section Outline

1. The Work Done by a Constant Force
2. Review of the Vector Dot Product
3. The Work Done by a Varying Force

Why do objects do what they do? Two ideas that help answer this question are force and linear momentum. However, both force and momentum have additional mathematical complications because they are vectors. Energy is another answer. Energy has the advantage of being a scalar quantity. As with momentum, we will be able to build a conservation law that is a very powerful tool for understanding why objects do what they do. In this section will start our journey toward the Law of Conservation of Energy by defining a concept called, "work."

1. The Work Done by a Constant Force

Think about the last time you had to push a car. Perhaps, the brake was on and you probably noticed that the force you exerted resulted in no acceleration of the car. Maybe you remembered to release the brake and the car slowly began to move. The point is that, while all forces in principle can cause acceleration, in some situations they just don't.



At the right is a block being pulled across a frictionless table by a constant force F, directed at an angle, θ, above the horizontal. The component of the force that is along the motion, $F_\parallel$, causes the block to accelerate, while the component perpendicular to the table, $F_\perp$, does not. It is generally true that forces along the direction of motion cause acceleration.

In analogy to developing linear momentum, where we defined a quantity called impulse which is the force multiplied by the time, we define a quantity called "work" which is the force multiplied by the distance. However, we only count the component of the force along the motion of the object, so the work done by a constant force is defined as,

$$\Delta W \equiv F_\parallel \Delta s\,,$$

where Δs is the distance the object moves.

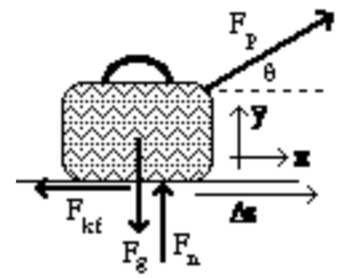
The units of work are the units of force times the units of distance so,

$$[\Delta W] = [F][s] = \mathrm{N}\cdot\mathrm{m}\,.$$

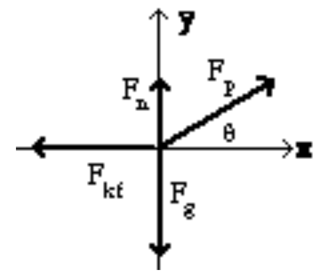
This unit has its own name, $1\mathrm{N\cdot m} \equiv 1$ Joule = 1J.

Example 19.1: A 50.0N force is exerted at 30.0° above horizontal on a 20.0kg suitcase. The suitcase moves at a constant speed for 4.00m. Find the work done by each force that acts on it.

Given: $F_p = 50.0\text{N}$, $\theta = 50.0°$, $m = 20.0\text{kg}$, and $\Delta s = 4.00\text{m}$.
Find: $W_p = ?$, $W_{kf} = ?$, $W_n = ?$, and $W_g = ?$



Free Body Diagram



To find the size of each force, apply the Second Law to each direction separately,

$$\Sigma F_x = ma_x \Rightarrow F_p \cos\theta - F_{kf} = 0 \Rightarrow F_{kf} = F_p \cos\theta,$$

$$\Sigma F_y = ma_y \Rightarrow F_n - F_g + F_p \sin\theta = 0 \Rightarrow F_n = F_g - F_p \sin\theta.$$

Using the mass/weight rule and plugging in the numbers,

$$F_g = mg = (20.0)(9.80) \Rightarrow F_g = 196\text{N},$$

$$F_{kf} = 50.0\cos 30.0° \Rightarrow F_{kf} = 43.3\text{N},$$

$$F_n = F_g - F_p \sin\theta = 196 - 50.0\sin 30.0° \Rightarrow F_n = 171\text{N}.$$

Using the definition of work,

$\Delta W \equiv F_{\parallel}\Delta s \Rightarrow W_p = (F_p \cos\theta)\Delta s = (43.3)(4) \Rightarrow \boxed{W_p = 173\text{J}}$.

Note that only the component along the motion counts.

$\Delta W \equiv F_{\parallel}\Delta s \Rightarrow W_{kf} = -F_{kf}\Delta s = (-43.3)(4) \Rightarrow \boxed{W_{kf} = -173\text{J}}$.

Since the friction and motion are opposite, negative work is done.

$\Delta W \equiv F_{\parallel}\Delta s \Rightarrow \boxed{W_n = 0}$ and $\boxed{W_g = 0}$.

The work done by the normal force and gravity is zero because the forces don't act along the motion.

The total work done by all the forces on the suitcase adds up to zero. This is because the suitcase doesn't accelerate. Now, let's examine the case when the suitcase accelerates.

Example 19.2: Repeat example 1 for the case where the coefficient of kinetic friction is 0.200.

Given: $F_p = 50.0\text{N}$, $\theta = 50.0°$, $m = 20.0\text{kg}$, $\mu = 0.200$ and $\Delta s = 4.00\text{m}$.
Find: $W_p = ?$, $W_{kf} = ?$, $W_n = ?$, and $W_g = ?$

The free body diagram is the same. The only difference is that the frictional force is smaller. Using the definition of the coefficient of friction,

$$\mu_k \equiv \frac{F_{kf}}{F_n} \Rightarrow F_{kf} = \mu_k F_n = (0.200)(171) \Rightarrow F_{kf} = 34.2N$$

The work done by friction is,

$$\Delta W \equiv F_{\parallel}\Delta s \Rightarrow W_{kf} = -F_{kf}\Delta s = (-34.2)(4) \Rightarrow \boxed{W_{kf} = -137J}.$$

The work done by the other three forces is the same as before,

$$\Delta W \equiv F_{\parallel}\Delta s \Rightarrow \boxed{W_p = 173J}, \boxed{W_n = 0} \text{ and } \boxed{W_g = 0}.$$

The total, or net, work done is,

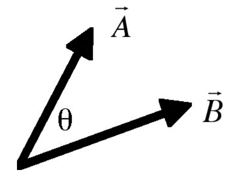
$W_{net} = W_p + W_{kf} + W_g + W_n = 173 - 137 + 0 + 0 = 35J$.

There is net work done when an object accelerates.

The definition, $\Delta W \equiv F_{\parallel}\Delta s$ is cumbersome and a bit sloppy. It is cleaner to write this using the dot product.

2. Review of the Vector Dot Product

There are two types of vector multiplication. One produces a scalar quantity and is called the "Dot" or "Scalar" product.



The Vector Dot Product $\vec{A} \bullet \vec{B} \equiv AB\cos\theta = A_xB_x + A_yB_y + A_zB_z$

> *Example 19.3: For the two vectors* $\vec{A} = 40.0\hat{i} + 20.0\hat{j}$ *and* $\vec{B} = 10.0\hat{i} + 30.0\hat{j}$ *find (a)their dot product and (b)the angle between them.*
>
> Given: $\vec{A} = 40.0\hat{i} + 20.0\hat{j}$ and $\vec{B} = 10.0\hat{i} + 30.0\hat{j}$.
> Find: $\vec{A} \bullet \vec{B} = ?$ and θ = ?
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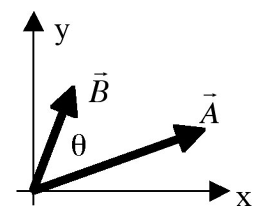
> 
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>
> (a)Using the equation for dot product,
> $\vec{A} \bullet \vec{B} = A_xB_x + A_yB_y = (40.0)(10.0) + (20.0)(30.0) \Rightarrow \boxed{\vec{A} \bullet \vec{B} = 1000}$.
> (b)Using the definition of the dot product,
> $$\vec{A} \bullet \vec{B} = AB\cos\theta \Rightarrow \theta = \arccos\frac{\vec{A} \bullet \vec{B}}{AB} = \arccos\frac{\vec{A} \bullet \vec{B}}{\sqrt{A_x^2 + A_y^2}\sqrt{B_x^2 + B_y^2}}$$
> Plugging in the numbers, $\theta = \arccos\frac{1000}{\sqrt{40^2 + 20^2}\sqrt{10^2 + 30^2}} \Rightarrow \boxed{\theta = 45°}$

The dot product selects the component of one vector along the other and multiplies them.

> *Example 19.4: For the two vectors* $\vec{A} = 40.0\hat{i} + 20.0\hat{j}$ *and* $\vec{B} = 10.0\hat{i} + 30.0\hat{j}$ *find the component* of $\vec{B}$ along $\vec{A}$.
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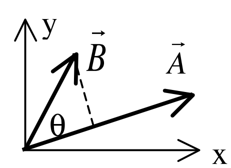
> 
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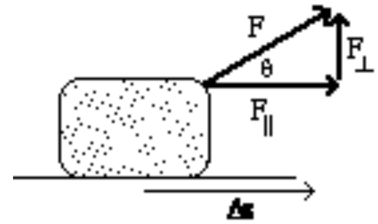
> Given: $\vec{A} = 40.0\hat{i} + 20.0\hat{j}$, $\vec{B} = 10.0\hat{i} + 30.0\hat{j}$, $\vec{A} \bullet \vec{B} = 1000$, and θ = 45.0°.
> Find: $B_{\parallel} = B\cos\theta = ?$
>
> The component of $\vec{B}$ along $\vec{A}$ is Bcosθ. Using the definition of the dot product,
> $$\vec{A} \bullet \vec{B} = AB\cos\theta \Rightarrow B\cos\theta = \frac{\vec{A} \bullet \vec{B}}{A} = \frac{\vec{A} \bullet \vec{B}}{\sqrt{A_x^2 + A_y^2}} = \frac{1000}{\sqrt{40^2 + 20^2}} \Rightarrow \boxed{B_{\parallel} = B\cos\theta = 22.4}.$$

The definition of work, $\Delta W \equiv F_{\parallel}\Delta s$ can now be written more cleanly using the dot product,

$$\Delta W \equiv \vec{F} \bullet \Delta\vec{s}\,.$$



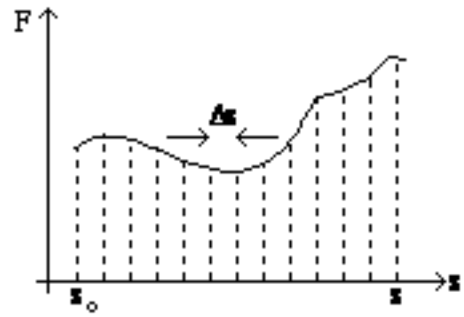
Comparing this to the old definition,

$$\Delta W \equiv \vec{F} \bullet \Delta\vec{s} = F\cos\theta\Delta s = F_{\parallel}\Delta s\,,$$

just like before.

3. The Work Done by a Varying Force

The next step in expanding the definition of work is to apply it to non-constant forces. The graph of a force that varies with distance is shown at the right. Since the definition of work can only be applied to constant forces, the interval between s_o and s must be broken down into smaller intervals, Δs, in which the force is nearly constant. In each of these small intervals, the definition of work applies, so the total work done is,



$$W = \Delta W_1 + \Delta W_2 + \cdots + \Delta W_N = F(s_o)\Delta x + F(s_o + \Delta s)\Delta s + \cdots + F(s)\Delta s\,.$$

The total work done is the sum of the areas of each little rectangle formed by the average force in each interval times the distance Δs. The sum of these areas is the total area under the curve. In other words, work can be defined as an integral.

Another way to look at this is to let the Δs's become so small that they become differentials,

$$\Delta W \equiv \vec{F} \bullet \Delta\vec{s} \to dW \equiv \vec{F} \bullet d\vec{s} \Rightarrow \int_0^W dW = \int_{s_o}^{s} \vec{F} \bullet d\vec{s} \Rightarrow W = \int_{s_o}^{s} \vec{F} \bullet d\vec{s}\,.$$

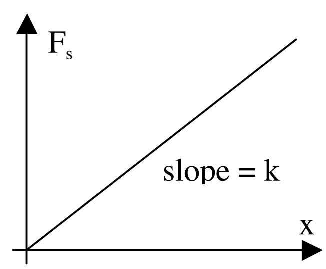
We will use this as the definition of work from now on,

The Definition of Work $W \equiv \int \vec{F} \bullet d\vec{s}$

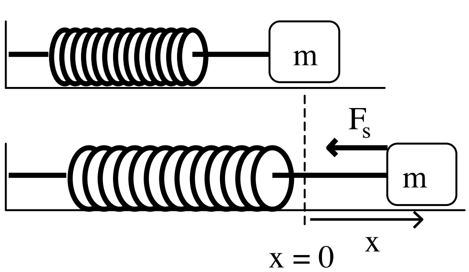
This definition of work includes all the previous definitions: if the force is constant, $\Delta W \equiv \vec{F} \bullet \Delta\vec{s}$, and the dot product just picks off the part of the force along the direction of motion, $\Delta W \equiv F_{\parallel}\Delta s$.

The next example looks at a non-constant force exerted by a spring. The force exerted by a spring grows as the spring is stretched. For an ideal spring, the force it exerts is linearly proportional to the stretch,

$$F_s \propto x \Rightarrow F_s = kx\,.$$

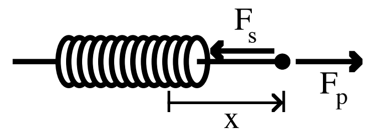


The constant of proportionality, k, is called the "spring constant" and it is the slope of the graph of the force versus the stretch. In the upper image at the left, the spring is relaxed and exerts no force on the mass. If the mass is moved a distance x to the right, as shown in the lower image, the spring exerts a force that is directed back to the left. A minus sign is need to keep track of the direction which leads us to



Hooke's Rule $\vec{F}_s = -k\vec{x}$.

Example 19.5: Using Hooke's Rule, find (a)the work you must do to stretch the spring a distance x and (b)the work done by the spring. Assume the stretching is done at constant speed.



Given: $\vec{F}_s = -k\vec{x}$
Find: W_p = ? and W_s = ?

To stretch the spring you must pull on it. Applying the Second Law to the end of the spring,

$$\Sigma F = ma \Rightarrow F_p - F_s = 0 \Rightarrow F_p = F_s.$$

The magnitude of both forces is the same since the velocity is constant. According to Hooke's Rule,

$$F_p = F_s = kx$$

(a)The work you must do, can be found from the definition of work,

$$W \equiv \int \vec{F} \bullet d\vec{s} \Rightarrow W_p = \int F_p dx = \int_0^x kxdx \Rightarrow \boxed{W_p = \tfrac{1}{2}kx^2}.$$

(b)The work done by the spring will be the same magnitude, but negative because the force exerted by the spring is opposite to the motion,

$$W \equiv \int \vec{F} \bullet d\vec{s} \Rightarrow W_s = -\int F_s dx = -\int_0^x kxdx \Rightarrow \boxed{W_s = -\tfrac{1}{2}kx^2}.$$

The total work done is equal to zero because the speed is constant.

Section 19 - Summary

Why do objects do what they do? In addition to the two previous ideas of force and linear momentum, we have begun to develop a third concept, energy. We are getting started by defining work.

$$\text{The Definition of Work } W \equiv \int \vec{F} \bullet d\vec{s}$$

where we use the

$$\text{Vector Dot Product } \vec{A} \bullet \vec{B} \equiv AB\cos\theta = A_x B_x + A_y B_y + A_z B_z.$$

The advantage of using the idea of work is that it is a scalar quantity as opposed to force that is a vector.

We examined the work done by the non-constant of a spring summarized by,

$$\text{Hooke's Rule } \vec{F}_s = -k\vec{x}.$$