Section 20 – The Work-Energy Theorem

Section Outline

1. The Work-Energy Theorem
2. Examples

Why do objects do what they do? Two ideas that help answer this question are force and linear momentum. However, both force and momentum have additional mathematical complications because they are vectors. Energy is another answer. Energy has the advantage of being a scalar quantity. As with momentum, we will be able to build a conservation law that is a very powerful tool for understanding why objects do what they do.

1. The Work-Energy Theorem

Remember pushing the car? If the brakes were on and the car didn't move, you did no work (at least according to our definition). However, if the can started accelerating due to your applied force, work was done. So, the total work done on an object must be related to the motion of an object.

When net work is done on an object, the object accelerates. This can be shown in general starting with the definition of work and applying the Second Law to some mass m that feels some forces,

$$W \equiv \int \vec{F} \bullet d\vec{s} \Rightarrow W_{net} = \int \Sigma\vec{F} \bullet d\vec{s} = \int m\vec{a} \bullet d\vec{s} = m\int \vec{a} \bullet d\vec{s}\,.$$

Using the definitions of acceleration and velocity,

$$W_{net} = m\int \frac{d\vec{v}}{dt} \bullet d\vec{s} = m\int d\vec{v} \bullet \frac{d\vec{s}}{dt} = m\int d\vec{v} \bullet \vec{v} = m\int v dv\,.$$

Assuming the object initially has a speed v_o and its final speed is v,

$$W_{net} = m\int_{v_o}^{v} v dv = m\left[\tfrac{1}{2}v^2\right]_{v_o}^{v} = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_o^2\,.$$

So, as we predicted, there is a relationship between the total work done and the change in speed. Another way to look at this result is, when net work is done on a object, it changes the value of this quantity, $\frac{1}{2}mv^2$, which is called the "kinetic energy."

The Definition of Kinetic Energy $K \equiv \frac{1}{2}mv^2$

The net work done can now be written as, $W_{net} = K - K_o$, which is the Work-Energy Theorem.

Work-Energy Theorem $W_{net} = \Delta K$

When net work is done on an object, its kinetic energy changes. Since work comes in Joules, so does kinetic energy.

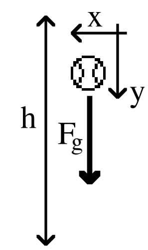
You should notice a striking resemblance to the Impulse-Linear Momentum Theorem and the definition of Impulse. This similarity is summarized in the chart below.

Impulse-Momentum Theorem $\Delta\vec{p} = \vec{J}$	Definition of Impulse $\vec{J} \equiv \int_{t_o}^{t} \vec{F}dt$
Work-Energy Theorem $\Delta K = W$	Definition of Work $W \equiv \int_{\vec{s}_o}^{\vec{s}} \vec{F} \bullet d\vec{s}$

The Impulse-Linear Momentum Theorem explains that a force acting over time creates an impulse that changes the linear momentum of an object, while the Work-Energy Theorem states that a force acting over a distance does work that changes the kinetic energy of an object.

2. Examples

Example 20.1: A ball is dropped from a height of 1.00m. Find the speed of the ball just before it strikes the ground.



Given: $y_o = 0$, $y = 1.00$m, and $v_o = 0$
Find: v = ?

We've done this problem before using kinematics, but let's examine it in terms of the work-energy theorem. The net work done on the falling ball is done by gravity. Using the definition of work,

$$W \equiv \int \vec{F} \bullet d\vec{s} \Rightarrow W_{net} = \int F_g\, dy = F_g \int_{y_o}^{y} dy = F_g(y - y_o) = mgy,$$

where the mass/weight rule has been used. Applying the work energy theorem and the definition of kinetic energy,

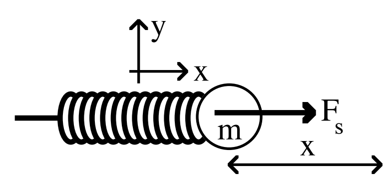
$$W_{net} = K - K_o \Rightarrow mgy = \tfrac{1}{2}mv^2 - \tfrac{1}{2}mv_o^2 \Rightarrow 2gy = v^2 - v_o^2 \Rightarrow v^2 = v_o^2 + 2gy,$$

which gives the kinematic equation we used when we did this problem before. Finishing the algebra and plugging in the numbers,

$$v = \sqrt{v_o^2 + 2gy} = \sqrt{0 + 2(9.80)(1.00)} \Rightarrow \boxed{v = 4.43\text{m/s}}.$$

Now let's attack a problem we could not have solved using kinematics.

Example 20.2: A spring (k=8000N/m) is compressed 5.00cm and a 50.0g ball is placed at the end. The spring is released. Find (a)the work done by the spring on the ball, (b)the final kinetic energy of the ball and (c)the speed that the ball leaves the spring.



Given: k = 8000N/m, x = 0.0500m, and m = 0.0500kg.
Find: W_s = ?, K = ?, and v = ?

(a)The work done by the spring can be found from the definition,

$$W \equiv \int \vec{F} \bullet d\vec{s} \Rightarrow W_s = \int F_s\, dx = \int_0^x kx\, dx = \tfrac{1}{2}kx^2.$$

Note that the work done by the spring is positive this time because the force it exerts is in the direction of motion. Plugging in the values,

$$W_s = \tfrac{1}{2}(8000)(0.0500)^2 \Rightarrow \boxed{W_s = 10.0\text{J}}.$$

(b)The kinetic energy can be found from the Work-Energy Theorem,

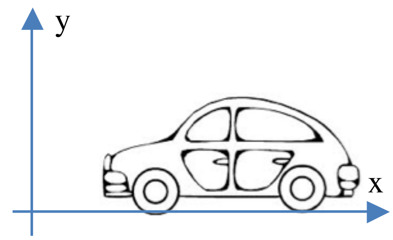
$$W_{net} = \Delta K \Rightarrow W_s = K - K_0 \Rightarrow K = W_s \Rightarrow \boxed{K = 10.0\text{J}}.$$

(c)Using the definition of kinetic energy,

$$K \equiv \tfrac{1}{2}mv^2 \Rightarrow v = \sqrt{\frac{2K}{m}} = \sqrt{\frac{2(10.0)}{0.0500}} \Rightarrow \boxed{v = 20.0m/s}.$$

This problem could have been done by using the Second Law and the definition of acceleration, then integrating to get the velocity. The work-energy method is much easier. The spring does net work on the mass causing it to speed up and increase its kinetic energy.

Example 20.3: An 800kg car 20m from an intersection is moving at 11m/s (25mph) when see the light turn red and apply the brakes. They come to rest just as they reach the intersection. Find (a)the change in the car's kinetic energy, (b)the work done by the road on the car, (c)the average force the road exerts on the car. (d)Suppose the car was traveling at twice the speed and the road exerts the same force, find the stopping distance.



Given: m = 80kg, x_o = 20m, x = 0, v_{o1} = 11m/s, v_{o2} = 22m/s, and v = 0

Find: ΔK = ?, W_r = ?, F_f = ?, and d = ?

(a)Using the definition of kinetic energy,

$$K \equiv \tfrac{1}{2}mv^2 \Rightarrow K_o = \tfrac{1}{2}mv_o^2 = \tfrac{1}{2}(800)(11)^2 \Rightarrow \boxed{K_o = 48.4kJ}.$$

Similarly,

$$K = \tfrac{1}{2}mv^2 = \tfrac{1}{2}(800)(0)^2 \Rightarrow \boxed{K_o = 0}.$$

(b)Applying the Work-Energy Theorem,

$$W_{net} = \Delta K \Rightarrow W_r = \Delta K = K - K_o = 0 - 48.4kJ \Rightarrow \boxed{W_r = -48.4kJ}.$$

Note that the work done is negative because the car is slowing down. This makes sense because the frictional force exerted by the road is opposite to the motion of the car.

(c)Using the definition of work,

$$W \equiv \int \vec{F} \bullet d\vec{s} \Rightarrow W_r = F_{fr}(x - x_o) \Rightarrow F_{fr} = \frac{W_r}{x - x_o} = \frac{-48.4x10^3}{0 - 20} \Rightarrow \boxed{F_{fr} = 2420N}.$$

(d)Now, the net work done is,

$$W_{net} = \Delta K \Rightarrow W_r = \Delta K = K - K_o = 0 - \tfrac{1}{2}mv_o^2 = -\tfrac{1}{2}(800)(22)^2 \Rightarrow \boxed{W_r = -194kJ}.$$

Four times higher! Using the definition of work,

$$W \equiv \int \vec{F} \bullet d\vec{s} \Rightarrow W_r = F_{fr}(x - x_o) \Rightarrow x - x_o = \frac{W_r}{F_{fr}} = \frac{-194x10^3}{2420} \Rightarrow \boxed{\Delta x = 80m}.$$

If you double your speed your stopping distance increases by a factor of four. Drive carefully!

<u>Section 20 - Summary</u>

Why do objects do what they do? In addition to the two previous ideas of force and linear The single most important idea in this section is the

$$\text{Work-Energy Theorem} \quad W_{net} = \Delta K \,,$$

where we have established

$$\text{The Definition of Kinetic Energy} \quad K \equiv \tfrac{1}{2} mv^2 \,.$$

The Work-Energy Theorem explains that when net work is done on an object, the object speeds up. The ideas in this section set the stage for the second conservation law, the Law of Conservation of Energy which is the topic of the next section.