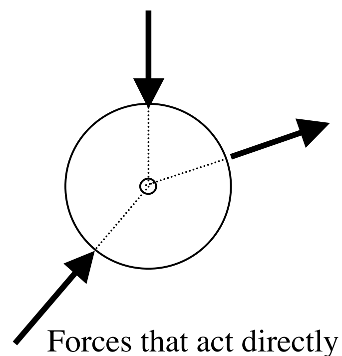
Section 26 – The Laws of Rotational Motion

What do objects do and why do they do it? They rotate and we have established the quantities needed to describe this motion. We now need to find the laws that explain it. Since we could find the rotational variables by analogy to the translational variables, we'll try to find the laws of rotational motion in analogy to the laws of translational motion.

Section Outline
1. The First Law of Rotational Motion
2. The Second Law of Rotational Motion

1. The First Law of Rotational Motion

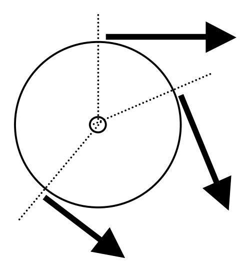
Newton's Laws of Motion work so well for translational motion that we should use them as a model of the laws for rotational motion. The idea here is to change the laws as little as possible, but still properly understand rotational motion.



Forces that act directly toward or away from the axle cause no rotation.

Newton's First Law
"Every object will move with a constant velocity unless a force acts on it."

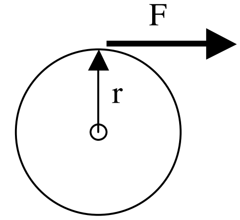
The First Law explained the natural state of motion and that force was the agent of change of motion. Just as the natural state of translational motion is any constant velocity, the natural state of rotational motion is any constant angular velocity. Think about a spinning top. It will keep spinning at a constant rate as long as frictional forces don't cause it to slow down. So it looks like we just need to add the word "angular" in front of "velocity." However, some forces don't cause changes in rotation.



Forces that don't act directly toward the axle do cause rotation.

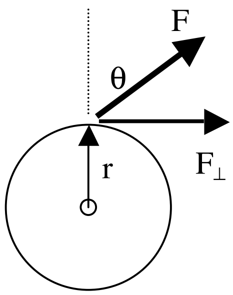
Think about the front wheel on your bike. Lift it off the ground so that it is free to spin. There are two forces acting on it, gravity downward and the force the axle exerts upward at the center. Neither causes rotation. You can even exert additional forces on the tire that won't cause it to rotate if you push directly toward the axle. If you push in any direction that doesn't point at the axle, the tire will begin to spin. This rotating effect of a force is called "torque." So, we must replace "force" in the First Law with "torque."

Newton's First Law *for Rotation*
"Every object will move with a constant *angular* velocity unless a ~~force~~ torque acts on it."



We need to get a bit more quantitative with torque. Think about a force that acts tangentially on the bike wheel, as shown at the right. The larger the force, the greater the change in angular velocity. However, the location of the force also matters. If the same force were exerted at half the distance from the center, it would cause half the change in angular velocity. So, we can write the torque as the product of the force and the radius at which it acts,

$$\tau = rF .$$



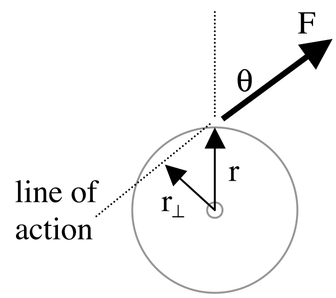
The force doesn't have to be applied perpendicular to the radius vector. Suppose the force acts at an angle θ to the radius vector as shown at the left. Now, only the component of the force that is perpendicular to the radius vector creates torque. This perpendicular component can be written in terms of the angle,

$$\tau = F_\perp r = (F\sin\theta)r\,.$$

We could associate the sinθ with the r instead of the F,

$$\tau = F_\perp r = (F\sin\theta)r = F(r\sin\theta) = Fr_\perp\,.$$

This is shown at the right. The line along which the force acts is referred to as the "line of action." The "moment arm" or "lever arm" is the perpendicular distance from the pivot point to the line of action of the force, $r_\perp$. In summary,



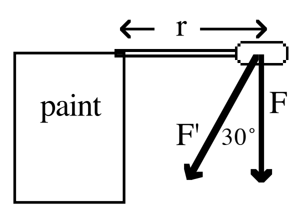
- forces with lines of action not passing through the axis of rotation (pivot) create torque.
- the torque grows as the force grows.
- the torque is proportional to the distance between the force and the pivot.
- any force can be resolved a component toward the pivot which creates no torque and a component perpendicular to the line to the pivot which causes torque.

We can encapsulate all of this using the mathematical idea of a cross product to define torque,

The Definition of Torque $\vec{\tau} \equiv \vec{r} \times \vec{F}$.

We'll worry about the direction of the torque vector a bit latter. For now we'll only deal with the magnitude.

Example 26.1: To open a paint can, a force of 100N is exerted downward on a 25.0cm long screwdriver. Find (a)the torque produced and (b)the force needed if it is exerted at an angle of 30.0° to the vertical.



Given: F = 100N, r = 0.250m, and θ = 30.0°
Find: τ = ? and F′ = ?

(a)Using the definition of torque and the fact that the moment arm and force are perpendicular,

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \Rightarrow \tau = rF = (0.250)(100) \Rightarrow \boxed{\tau = 25.0\,\text{N}\cdot\text{m}}.$$

(b)When the force and moment arm are not perpendicular, it is only the perpendicular part of the force that counts,

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \Rightarrow \tau = rF'\cos 30^\circ \Rightarrow F' = \frac{\tau}{r\cos 30^\circ} = \frac{25.0}{0.25\cos 30^\circ} \Rightarrow \boxed{F' = 115\text{N}}.$$

2. The Second Law of Rotational Motion

The Second Law told us how much force was required to change the velocity at a given rate and defined the idea of inertia.

Newton's Second Law

"Acceleration of an object is directly proportional to the net force acting on it and inversely proportional to its mass (inertia)."

We definitely need to replace the "acceleration" with "angular acceleration" and "force" with "torque." Instead of "mass" we need to understand the inertia associated with rotation. It is called "rotational inertia."

Newton's Second Law *for Rotation*

"*Angular* Acceleration of an object is directly proportional to the net ~~force~~ *torque* acting on it and inversely proportional to its ~~mass~~ *rotational inertia*."

Mathematically we just need to change $\Sigma F = ma$ to $\Sigma\tau = I\alpha$ where I is the rotational inertia.

The Second Law for Rotation $\Sigma\tau = I\alpha$

What is the rotational inertia? What does it depend on? How do you calculate it?

Consider a solid object pivoted at a certain point having an angular acceleration, α, as shown at the right. For any small chunk of its mass, dm, the force, dF, on this chunk must be perpendicular to the vector from the pivot. Otherwise, this chunk would accelerate toward the pivot. The torque, dτ, on this chunk of mass is given by the definition,

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \Rightarrow d\tau = rdF .$$

Using the Second Law for Translation and the relationship between tangential acceleration and angular acceleration,

$$d\tau = rdF = radm = \alpha r^2 dm .$$

Integrating over the entire object,

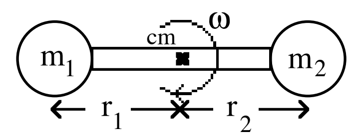
$$\int d\tau = \int \alpha r^2 dm \Rightarrow \tau = \alpha \int r^2 dm .$$

Comparing this result with the Second Law for Rotation gives the expression that will serve as the definition of rotational inertia.

The Definition of Rotational Inertia $I \equiv \int r^2 dm$

Regular inertia, the resistance objects offer to changes in translation, depends only on the mass of an object. Rotational inertia, the resistance objects offer to changes in rotation, depends not only on the mass of the object, but also on the shape.

Example 26.2: A dumb bell consisting of two 10.0kg masses 50.0cm apart separated by a light stick. It is rotating about its center of mass. Find the torque required to slow it uniformly from 5.00rpm to rest in 3.00s.



Given: $m_1 = m_2 = 10.0\text{kg}$, $r_1 = r_2 = 0.250\text{m}$, $\omega_o = 0$, $\omega = 5.00\text{rpm}$, and $\Delta t = 3.00\text{s}$
Find: $\tau = ?$

In order to apply the Second Law for Rotation, we need the rotational inertia and the angular acceleration. Since the masses are small compared to their distance from the axis of rotation, the integral in the definition of rotational inertia becomes a sum,

$$I \equiv \int r^2 dm \Rightarrow I = \sum_i m_i r_i^2 .$$

Putting in the numbers, $I = m_1 r_1^2 + m_2 r_2^2 = (10)(0.25)^2 + (10)(0.25)^2 = 1.25\text{kg}\cdot\text{m}^2$.
Now we can use the definition of angular acceleration,

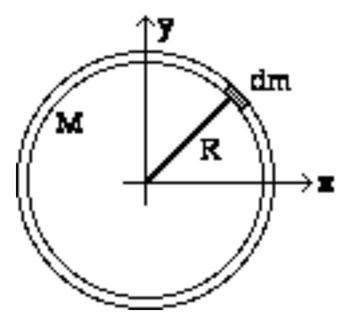
$$\alpha \equiv \frac{d\omega}{dt} \Rightarrow \alpha = \frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_o}{\Delta t} = \frac{(5.00\,\frac{\text{rev}}{\text{min}})(\frac{\text{min}}{60\,\text{s}})(\frac{2\pi\text{rad}}{\text{rev}}) - 0}{3.00} = 0.175\text{rad/s}^2 .$$

Finally, using the Second Law,

$$\Sigma\tau = I\alpha \Rightarrow \tau = I\alpha = (1.25)(0.175) \Rightarrow \boxed{\tau = 0.219\text{N}\cdot\text{m}}.$$

An "extended object" is one where the distance from the rotation axis to the mass elements is not large compared to the size of the object. In this case, the rotational inertia must be found by integration.

Example 26.3: (a)Find the rotational inertia of a uniform ring of radius, R, and mass, M. (b)Repeat for a disk.



Given: M and R
Find: I_{ring} and I_{disk}

Use the definition of rotational inertia keeping in mind that all the dm's have the same r = R,

$$I \equiv \int r^2 dm = \int R^2 dm = R^2 \int dm = R^2 M .$$

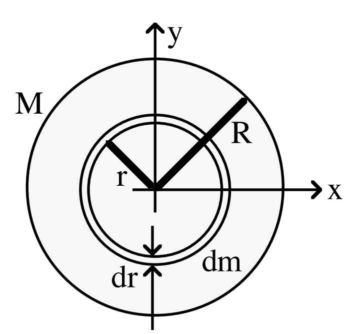
The sum of all the dm's is M. The rotational inertia for a ring is then,

$$\boxed{I_{ring} = MR^2}.$$

For a disk, the dm is an infinitely thin ring of radius r. The rotational inertia of this ring is dI given by the expression we just found,

$$dI = r^2 dm .$$

Adding these up over all the rings will give the rotational inertia of the disk,

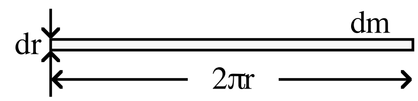


$$\int_0^{I_{disk}} dI = \int_0^R r^2 dm \Rightarrow I_{disk} = \int_0^R r^2 dm .$$

The mass, dm, in each ring depends on the radius. Assuming the disk is uniform, the mass is distributed in proportion to the surface area so,

$$\frac{dm}{M} = \frac{dA}{A}.$$

The area of the disk is $A = \pi R^2$. The dA is the area of the thin ring. At right is a sketch of this ring laid out in a straight line of thickness dr and length $2\pi r$. It has an area of $dA = 2\pi r dr$. So,

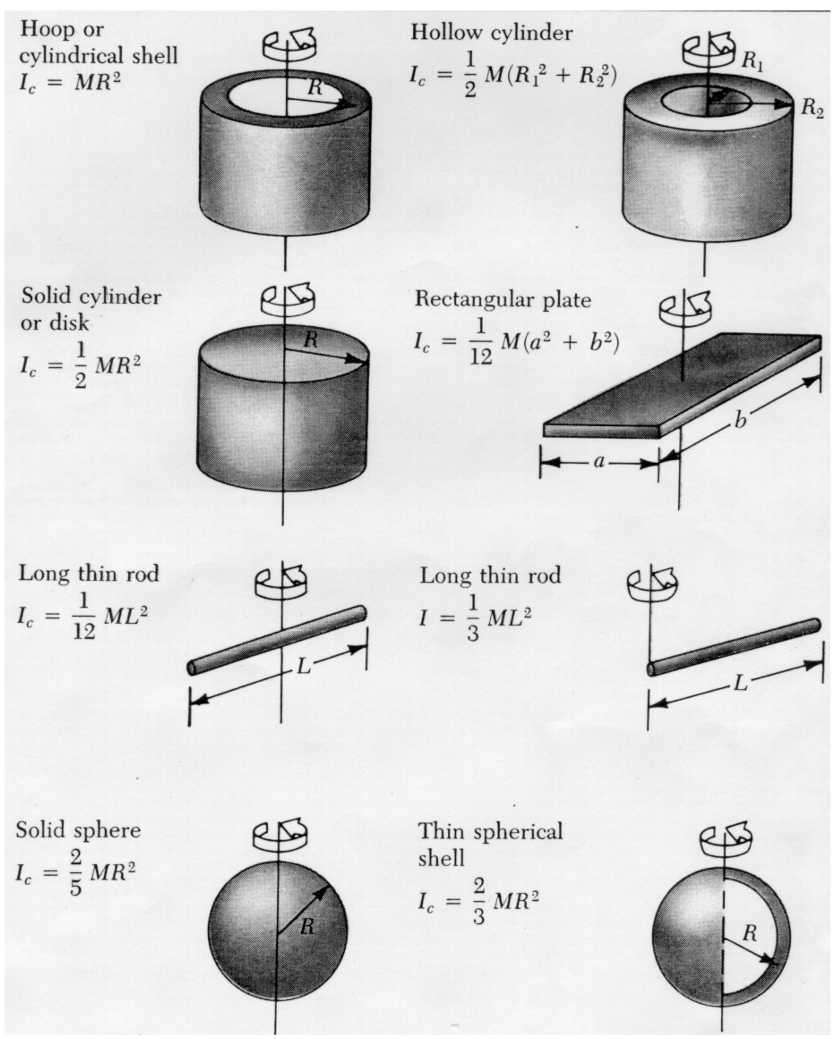


$$\frac{dm}{M} = \frac{2\pi r dr}{\pi R^2} \Rightarrow dm = \frac{2M}{R^2} r dr .$$

Now the integration can be done,

$$I_{disk} = \int_0^R r^2 \frac{2M}{R^2} r dr = \frac{2M}{R^2} \int_0^R r^3 dr = \frac{2M}{R^2}\left(\frac{R^4}{4}\right) \Rightarrow \boxed{I_{disk} = \tfrac{1}{2} MR^2}.$$

The rotational inertia of objects depends on their shape. Here is a table listing the equation for the rotational inertia of some objects:



Section Summary

We now have some understanding of why objects rotate the way they do. We built the laws of rotational motion in analogy to Newton's Laws of Motion for translation,

Newton's First Law *for Rotation*
"Every object will move with a constant angular velocity unless a torque acts on it."

Newton's Second Law *for Rotation*
"Angular acceleration of an object is directly proportional to the net torque acting on it and inversely proportional to its rotational inertia."

The First Law led us to define torque as the rotating effect of a force,

$$\text{The Definition of Torque } \vec{\tau} \equiv \vec{r} \times \vec{F} \ (\tau = F_{\perp}r = Fr_{\perp}).$$

The Second Law was rewritten mathematically as,

$$\text{The Second Law for Rotation } \Sigma\tau = I\alpha,$$

where we defined the rotational inertia as,

$$\text{The Definition of Rotational Inertia } I \equiv \int r^2 dm .$$