Section 27 – Using Torque and Rotational Kinetic Energy

We intend to understand why objects rotate like they do using the Laws of Rotational Motion, so we better apply them a few times to get s sense of how they work. In addition, we'll examine rotational kinetic energy, which will allow us to use the Law of Conservation of Energy for rotating objects.

Section Outline
1. Using the Laws of Rotational Motion
2. Rotational Kinetic Energy

1. Using the Laws of Rotational Motion

As a reminder, here are the laws again.

Newton's First Law *for Rotation*
"Every object will move with a constant angular velocity unless a torque acts on it."

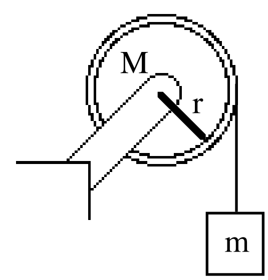
This led us to the definition of torque $\vec{\tau} = \vec{r} \times \vec{F}$.

Newton's Second Law *for Rotation*
"Angular acceleration of an object is directly proportional to the net torque acting on it and inversely proportional to its rotational inertia."

Written mathematically as $\Sigma\tau = I\alpha$, leading us to the definition of rotational inertia $I \equiv \int r^2\, dm$.

Now let's build our understanding by applying these laws.

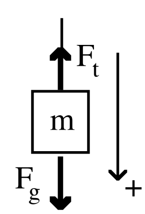
Example 27.1: A 500g mass hangs from a string that is wrapped around a 2.00kg solid cylinder 10.0cm in diameter as shown at the right. Find the acceleration of the hanging mass.



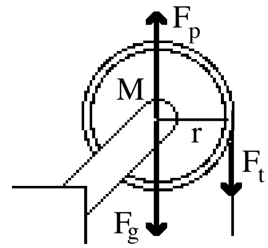
Given: m = 0.500kg, M = 2.00kg, and r = 0.0500m.
Find: a = ?

It is best to attack these types of problems by looking at separate parts of the system. Considering just the forces on the hanging weight and applying Newton's Second Law,

$$\Sigma F = ma \Rightarrow F_g - F_t = ma \Rightarrow mg - F_t = ma \, .$$



We have chosen the coordinates so down is positive, since that is the way things are moving.



Now consider the forces acting on the cylinder. Earth pulls downward as does the tension in the string. The pivot must exert an upward force to keep the disk from translating. To keep the coordinate systems consistent, all torques causing the hanging mass to drop are considered positive.

Note that the only force causing a torque about the center of the pulley is the tension. Applying the Second Law for Rotation,

$$\Sigma\tau = I\alpha \Rightarrow rF_t = I\alpha \Rightarrow F_t = \frac{I\alpha}{r}.$$

Substituting into the equation for the hanging mass,

$$mg - \frac{I\alpha}{r} = ma.$$

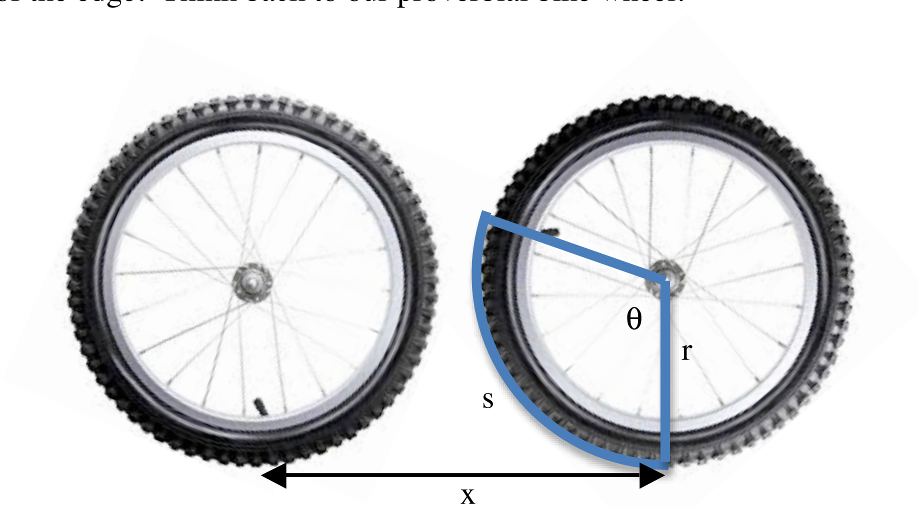
The rotational inertia of a disk is $I_{disk} = \frac{1}{2}MR^2$ and since the edge of the cylinder has a tangential acceleration of a which is related to the angular acceleration by $a_t = r\alpha$,

$$mg - \frac{\frac{1}{2}Mr^2\left(\frac{a}{r}\right)}{r} = ma \Rightarrow mg - \frac{1}{2}Ma = ma \Rightarrow \left(m + \frac{1}{2}M\right)a = mg \Rightarrow a = \frac{m}{m + \frac{1}{2}M}g.$$

Putting the numbers in,

$$a = \frac{0.500}{0.500 + \frac{1}{2}(2.00)}(9.80) \Rightarrow \boxed{a = 3.27\,\text{m/s}^2}.$$

Objects that roll without slipping have a special relationship between the motion of their center of mass and the motion of the edge. Think back to our proverbial bike wheel.



The distance traveled by the center of the rolling wheel, x, must be exactly equal to the arc length covered by the edge, s. After all, if a wheel rolls through one complete rotation, it must have moved forward a distance equal to its circumference. Since both things happen in the same amount of time, the tangential velocity must equal the velocity of the center-of-mass and the tangential acceleration must equal the acceleration of the center-of-mass. Mathematically, the derivation goes like this,

$$x = s = r\theta \Rightarrow \frac{dx}{dt} = r\frac{d\theta}{dt} \Rightarrow v_{cm} = r\omega.$$

Continuing on to the acceleration,

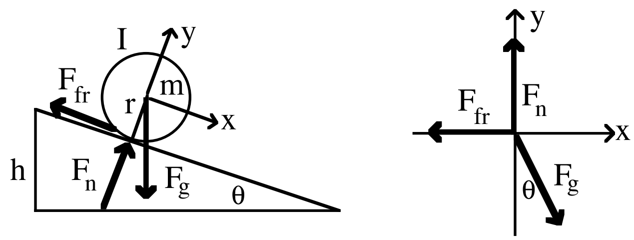
$$v_{cm} = r\omega \Rightarrow \frac{dv_{cm}}{dt} = r\frac{d\omega}{dt} \Rightarrow a_{cm} = r\alpha \,.$$

In summary, objects that roll without slipping have a special relationship between the motion of their center of mass and their rotational motion.

Rolling without slipping requires $x_{cm} = r\theta$, $v_{cm} = r\omega$, and $a_{cm} = r\alpha$.

Example 27.2: Find the acceleration of an object with mass, m, radius, r, and rotational inertia, I, rolls along an incline.

Given: m, r, and I.
Find: a = ? and v = ?



The sketch at the right shows the forces acting on the object and the free body diagram. Note the frictional force must be up the hill so it provides the correct torque to speed up the rotation if the object if it is going downward and to slow the rotation if it is going upward. Applying the Second Laws for Translation and Rotation,

$$\Sigma F_x = ma_x \Rightarrow F_g \sin\theta - F_{fr} = ma \Rightarrow mg\sin\theta - F_{fr} = ma \quad (1)$$

$$\Sigma F_y = ma_y \Rightarrow F_n - F_g\cos\theta = 0 \Rightarrow F_n = mg\cos\theta \quad (2)$$

$$\Sigma\tau = I\alpha \Rightarrow F_{fr}r = I\alpha \Rightarrow F_{fr} = \frac{I\alpha}{r} \quad (3)$$

Substituting eq. 3 into eq. 1,

$$mg\sin\theta - \frac{I\alpha}{r} = ma \,.$$

Since the object rolls without slipping $a = r\alpha \Rightarrow \alpha = \frac{a}{r}$. Substituting,

$$mg\sin\theta - \frac{I\left(\frac{a}{r}\right)}{r} = ma \Rightarrow mg\sin\theta - \frac{Ia}{r^2} = ma \Rightarrow mr^2 g\sin\theta = Ia + mr^2 a = a\left(I + mr^2\right) \Rightarrow$$

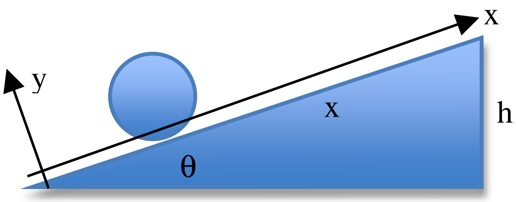
$$\boxed{a = \frac{mr^2}{I + mr^2} g\sin\theta}.$$

Note that if the object doesn't roll (I = 0), then the answer is $a = g\sin\theta$ which is the answer we got for a block sliding down a plane without friction. The equation for the acceleration shows us that the larger the rotational inertia, the smaller the acceleration. This must be true because the harder an object is to rotate the less rapidly it should speed up.

Let's continue on to find the resulting motion given these forces and torques.

Example 27.3: An object with mass, m, radius, r, and rotational inertia, I, rolls along horizontal ground with a speed, v, when it reaches the incline of the previous example. Find (a)the distance along the incline, and (b) the height it reaches when it momentarily comes to rest.

Given: $x_o = 0$, $v_o = v$, $v = 0$, and
$a = -\frac{mr^2}{I + mr^2} g \sin\theta$.
Find: x = ? and h = ?



Note, we have switched coordinates from the last example because we can keep the position and velocities positive, although the acceleration is now negative.
(a)Since this acceleration is constant, the speed at the bottom can be found using the kinematic without the time,

$$v^2 = v_o^2 + 2a(x - x_o) \Rightarrow 0 = v^2 - 2\left(\tfrac{mr^2}{I+mr^2} g \sin\theta\right)(x - 0) \Rightarrow \boxed{x = \tfrac{v^2}{2g\sin\theta}\left(\tfrac{I+mr^2}{mr^2}\right)}.$$

(b)Using trigonometry,

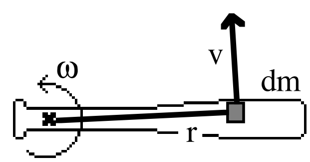
$$h = x\sin\theta = \tfrac{v^2}{2g\sin\theta}\left(\tfrac{I+mr^2}{mr^2}\right)\sin\theta \Rightarrow \boxed{h = \tfrac{v^2}{2g}\left(\tfrac{I+mr^2}{mr^2}\right)}.$$

Again, note that if the object doesn't roll, then the answer is the expected result, $h = \frac{v^2}{2g}$.

To summarize, torques can be used to explain the rotation about the center of mass, while forces describe the translation of the center of mass.

2. Rotational Kinetic Energy

As with many problems, like the last one, using the Law of Conservation of Energy is easier than using forces and torques. The trouble is that we have to have a way of keeping track on the kinetic energy associated with the rotation of the object. Basically, all that needs to be done is to add up the kinetic energy, dK, of each mass element, dm,



$$dK = \tfrac{1}{2}(dm)v^2 = \tfrac{1}{2}v^2 dm .$$

This velocity is tangential so it is related to the angular velocity by $v = r\omega$. Now,

$$dK = \tfrac{1}{2}\omega^2 r^2 dm \Rightarrow \int dK = \tfrac{1}{2}\omega^2 \int r^2 dm .$$

The left hand side is the total rotational kinetic energy and the integral on the right hand side is the rotational inertia.

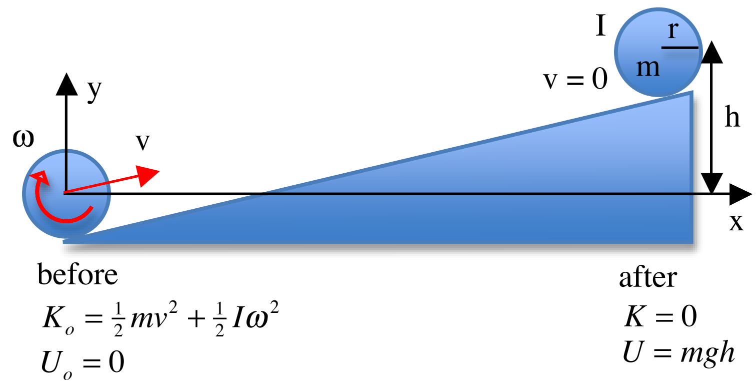
The Rotational Kinetic Energy $K = \frac{1}{2} I\omega^2$

Since the translational kinetic energy is one-half the inertia times the translational velocity squared, we might have guessed the rotational kinetic energy is one-half the rotational inertia times the rotational velocity squared.

Example 27.4: Repeat the last example using energy conservation to find the maximum height of the rolling object.

Given: m, I, r, and v.
Find: h = ?

There are two terms in the initial kinetic energy. The first term is the kinetic energy of the center of mass and the second term is the kinetic energy of rotation about the center of mass. Using the Law of Conservation of Energy,



$$\Delta U + \Delta K = 0 \Rightarrow \left[mgh - 0\right] + \left[0 - \left(\tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2\right)\right] \Rightarrow mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\omega^2 .$$

The velocity of the center of mass is related to the angular velocity because the object rolls without slipping,

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}I\left(\tfrac{v}{r}\right)^2 \Rightarrow mr^2gh = \tfrac{1}{2}mr^2v^2 + \tfrac{1}{2}Iv^2 = \left(\tfrac{1}{2}mr^2 + \tfrac{1}{2}I\right)v^2 \Rightarrow \boxed{h = \frac{v^2}{2g}\left(\tfrac{I+mr^2}{mr^2}\right)}.$$

We get the same result as before but using conservation of energy is much easier than working through the forces and torques.

When objects move due to the gravitational force, the mass almost never matter. What's going on here.

Example 27.4: The height in the last two examples appears to depend on the mass of the object. Show that the mass actually cancels out.

Recall that the rotational inertia is always some fraction times the quantity mr^2. Using f to represent the fraction, $I = fmr^2$. Substituting into the equation for the velocity,

$$h = \frac{v^2}{2g}\left(\tfrac{fmr^2 + mr^2}{mr^2}\right) = \frac{v^2}{2g}\left(\tfrac{f+1}{1}\right) \Rightarrow \boxed{h = \frac{v^2}{2g}(f+1)}.$$

The mass and the radius cancel out. So, as we suspected, the final speed doesn't depend upon the mass. It does depend upon the shape because f is determined by the fraction in the equation for rotational inertia. Note that objects with a bigger value of f go higher up the incline as they must since they have more rotational kinetic energy at the bottom.

Section Summary

We built our understanding of the Laws of Rotational Motion by apply them to some examples. The forces explain the motion of the center-of-mass while the torques explain the rotational motion.

When an object rolls without slipping there are special relationships between the motion of the center-of-mass and the rotational motion summarized by $x_{cm} = r\theta$, $v_{cm} = r\omega$, and $a_{cm} = r\alpha$.

Finally, recalling energy is often easier to deal with than forces and torques, we established the equation for,

The Rotational Kinetic Energy $K = \frac{1}{2} I\omega^2$.

Now we can use the Law of Conservation of Energy by including the rotational kinetic energy.

To complete our understanding of rotational motion, we need to continue to build upon our understanding of translational motion by looking into the rotational analog of linear momentum. In the process, we will need to look into the vector nature of torque.