Section 28 – Torque as a Vector

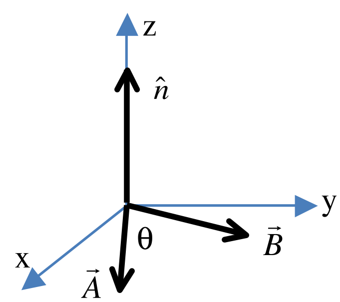
What do objects do and why do they do it? Objects rotate as well as translate. We can describe this motion with rotational kinematics and explain it in terms of torques and rotational kinetic energy. However, the concept of torque needs to be expanded because it as a vector. After a review of the vector cross product, the torque vector will be redefined in terms of the cross product.

Section Outline
1. The Vector Cross Product
2. Torque as a Vector

1. The Vector Cross Product

There are two basic types of vector multiplication. One produces a scalar quantity and is called the dot product. The second type produces a vector and is called the "cross" or "vector" product. Recall the dot product was related to the component of one vector along another. It was ideal for expressing the concept of work because we needed the component of the force along the motion. For torque, we need the component of the force perpendicular to the lever arm. The cross product is related to the component of one vector perpendicular to another. So, the cross product is just what we need.

Let's review the cross product. The cross product between two vectors A and B defined as,



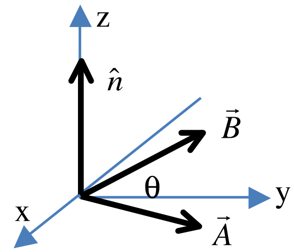
$$\text{Cross Product: } \vec{A}\times\vec{B} \equiv AB\sin\theta\hat{n}$$

where θ is the angle between them and $\hat{n}$ is a vector that is perpendicular to both A and B as shown at the right.

There are two equivalent ways to express the cross product in terms of the component of the vectors A and B given by,

$$\vec{A}\times\vec{B} = \left(A_yB_z - A_zB_y\right)\hat{i} + \left(A_zB_x - A_xB_z\right)\hat{j} + \left(A_xB_y - A_yB_x\right)\hat{k} \text{ or } \vec{A}\times\vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Example 28.1: For the two vectors $\vec{A} = 10.0\hat{i} + 20.0\hat{j}$ *and* $\vec{B} = -30.0\hat{i} + 20.0\hat{j}$ *find (a)their cross product and (b)the angle between them.*



Given: $\vec{A} = 10.0\hat{i} + 20.0\hat{j}$ and $\vec{B} = -30.0\hat{i} + 20.0\hat{j}$.
Find: $\vec{A} \times \vec{B} = ?$ and θ = ?.

(a)Using the equation for the cross product,

$$\vec{A} \times \vec{B} = \left(A_y B_z - A_z B_y\right)\hat{i} + \left(A_z B_x - A_x B_z\right)\hat{j} + \left(A_x B_y - A_y B_x\right)\hat{k}$$

$$\vec{A} \times \vec{B} = \left[(20)(0) - (0)(20)\right]\hat{i} + \left[(0)(-30) - (10)(0)\right]\hat{j} + \left[(10)(20) - (20)(-30)\right]\hat{k}$$

$\boxed{\vec{A} \times \vec{B} = 800\hat{k}}$.

If you prefer the matrix,

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 10 & 20 & 0 \\ -30 & 20 & 0 \end{vmatrix} = 0\hat{i} + 0\hat{j} + \left[(10)(20) - (20)(-30)\right]\hat{k} \Rightarrow \boxed{\vec{A} \times \vec{B} = 800\hat{k}}.$$

The cross product points in the z-direction as it should because both A and B are in the x-y plane to which the z-axis is perpendicular.

(b)Using the definition of the cross product,

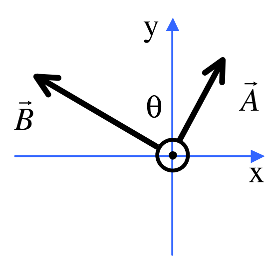
$$\left|\vec{A} \times \vec{B}\right| = AB\sin\theta \Rightarrow \theta = \arcsin\frac{\left|\vec{A} \times \vec{B}\right|}{AB} = \arcsin\frac{\left|\vec{A} \times \vec{B}\right|}{\sqrt{A_x^2 + A_y^2}\sqrt{B_x^2 + B_y^2}}$$

Plugging in the numbers,

$$\theta = \arcsin\frac{800}{\sqrt{10^2 + 20^2}\sqrt{(-30)^2 + 20^2}} \Rightarrow \boxed{\theta = 82.9^\circ}$$

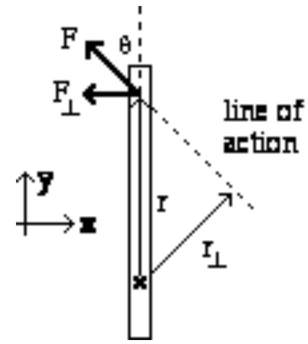
Another common way of indicating the direction of vectors that point in or out of a page is:

$\odot$ = out of the page
$\otimes$ = into the page



Perhaps these symbols come from looking directly at the point of the vectors arrow as is comes out of the page and seeing the tail feathers of the arrow when it points into the page. Using this notation, a sketch of the answer to the last example would look like the image at the right.

2. Torque as a Vector



In the last chapter we found the torque by taking the part of the force that is perpendicular to the radius vector and multiplying by r,

$$\tau = F_\perp r = (F\sin\theta)r,$$

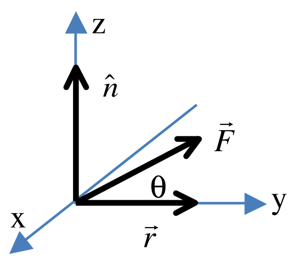
or by taking the part of the radius vector that is perpendicular to the line of action of the force (the lever arm) and multiplying by all of the force,

$$\tau = Fr_\perp = Fr\sin\theta.$$

In either case, it looks so much like a cross product that we will redefined torque as,

$$\vec{\tau} \equiv \vec{r} \times \vec{F}.$$

As a result, the torque points out of the paper or along the z-axis in this case as shown on the right below.

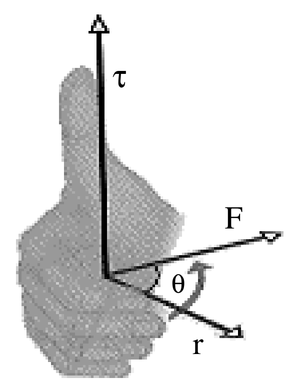


This may be bothersome at first. However, when you think about it, which way would you point a vector that is describing a rotation in the x-y plane. If it pointed in the x-y plane, it would have to be moving. This is problematic when the torque vector is constant. The only direction it could point is perpendicular to the x-y plane, along the z-direction.

The decision as to whether it points in the positive or negative z-direction is determined by the convention called the "right hand rule."

The Right Hand Rule:

- point your fingers along $\vec{r}$.
- rotate your hand so that you can bend your fingers the shorter way around toward $\vec{F}$.
- your thumb points along $\vec{\tau}$.



Example 28.2: A 200g meterstick is pointed northward and held horizontally at one end. Find the torque due to gravity about this end.

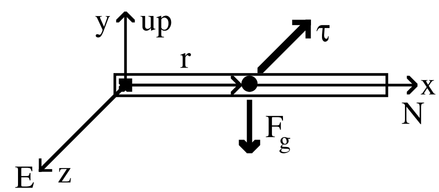
Given: m = 0.200kg and ℓ = 1.00m.
Find: τ = ?

Using the definition of torque,

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \Rightarrow \tau = rF_g \sin 90° = \tfrac{\ell}{2} mg = \tfrac{1}{2}(0.2)(9.8) \Rightarrow$$

$$\boxed{\tau = 0.980 N \cdot m}.$$

Using the right hand rule, the direction of the torque is $\boxed{\text{westward}}$ or in the negative z-direction.



Section Summary

We reviewed the mathematics of the vector cross product defined as,

$$\vec{A} \times \vec{B} \equiv AB \sin\theta \hat{n}$$

The cross product can be calculated mathematically using,

$$\vec{A} \times \vec{B} = \left(A_y B_z - A_z B_y\right)\hat{i} + \left(A_z B_x - A_x B_z\right)\hat{j} + \left(A_x B_y - A_y B_x\right)\hat{k} \text{ or } \vec{A} \times \vec{B} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}.$$

The mathematics of the cross product led to a deeper understanding of the vector nature of torque redefined as,

$$\text{Definition of Torque } \vec{\tau} \equiv \vec{r} \times \vec{F}.$$