Section 29 – Torque and Angular Momentum

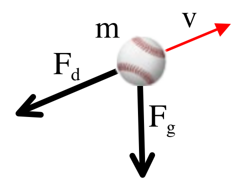
What do objects do and why do they do it? We are in the process of assigning vectors to the quantities, like torque, that explain rotational motion. When we examined force in detail we found a quantity called linear momentum and as a result found a powerful law, Conservation of Linear Momentum. In this section we'll look more carefully at torque and find a quantity called "angular momentum." Using the analogy to the Law of Conservation of Linear Momentum, we will develop a new law, the Conservation of Angular Momentum.

Section Outline

1. A Review of Force and Linear Momentum
2. Torque and Angular Momentum
3. The Definition of Angular Momentum

1. A Review of Force and Linear Momentum

The Second Law tells us the motion of a baseball through the air is determined by the forces of gravity and air resistance. That is, the net force is related to the change in velocity.



$$\Sigma\vec{F} = m\vec{a} \Rightarrow \Sigma\vec{F} = m\frac{d\vec{v}}{dt}$$

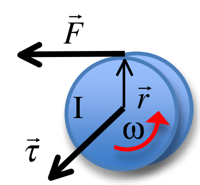
Recall that we rewrote Newton's Second Law in such a way that it led to the definition of linear momentum,

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \Sigma\vec{F} = m\frac{d\vec{v}}{dt} \Rightarrow \Sigma\vec{F} = \frac{d(m\vec{v})}{dt} \Rightarrow \Sigma\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow \vec{p} \equiv m\vec{v}\,.$$

So, we can interpret the law to say the net force on the ball causes the linear momentum to change.

2. Torque and Angular Momentum

We can take a similar approach using the definition of angular acceleration and Newton's Second Law for Rotation. Consider a wheel of rotational inertia, I, with a force exert tangentially along the edge resulting in a torque out of the page as shown at the right. Writing the Second Law for Rotation,



$$\Sigma\vec{\tau} = I\vec{\alpha} \Rightarrow \Sigma\vec{\tau} = I\frac{d\vec{\omega}}{dt} \Rightarrow \Sigma\vec{\tau} = \frac{d(I\vec{\omega})}{dt}\,.$$

The net torque causes changes in the quantity Iω called the angular momentum of a rigid body,

The Angular Momentum of a Rigid Body $\vec{L} = I\vec{\omega}$

Now,

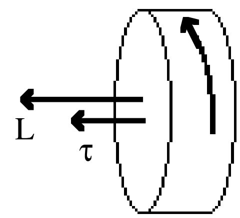
$$\Sigma\vec{\tau} = \frac{d(I\vec{\omega})}{dt} \Rightarrow \Sigma\vec{\tau} = \frac{d\vec{L}}{dt}\,.$$

Now, we can think of torque as the agent that changes angular momentum in analogy to the way we think of force as the agent of change for linear momentum.

Newton's Second Law for Rotation $\Sigma\vec{\tau} = \dfrac{d\vec{L}}{dt}$

Example 29.1: A 2.00kg grinding wheel has a radius of 10.0cm. It accelerates from rest to 300rpm in 4.00s. Find (a)the final angular momentum and (b)the net torque required.

Given: $m = 2.00\text{kg}$, $r = 0.100\text{m}$, $\omega_o = 0$, $\omega = 300\text{rpm} = 31.4\text{rad/s}$, and $\Delta t = 4.00\text{s}$.
Find: $L = ?$ and $\tau = ?$



(a)The grinding wheel is a disk so its rotational inertia is,

$$I = \tfrac{1}{2}mr^2 = \tfrac{1}{2}(2.00)(0.100)^2 = 0.0100\,\text{kg}\cdot\text{m}^2.$$

The angular momentum of the rigid body is,

$$\vec{L} = I\vec{\omega} \Rightarrow L = (0.01)(31.4) \Rightarrow \boxed{L = 0.314\text{kg}\cdot\text{m}^2/\text{s}}.$$

(b)Using the Second Law for Rotation,

$$\Sigma\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow \tau = \frac{\Delta L}{\Delta t} = \frac{L - L_o}{t} = \frac{L}{t} = \frac{0.314}{4.00} \Rightarrow \boxed{\tau = 0.0785\,\text{N}\cdot\text{m}}.$$

The change in angular momentum is caused by this torque. The directions of the vectors are shown in the sketch.

3. **The Definition of Angular Momentum**

There is a relationship between linear momentum and angular momentum that is the basis for the definition of angular momentum. This relationship comes from combining the definition of torque with the Second Laws for rotation and translation,

$$\vec{\tau} \equiv \vec{r} \times \vec{F} \Rightarrow \frac{d\vec{L}}{dt} = \vec{r} \times \frac{d\vec{p}}{dt}$$

While you would probably never think to do it, you can add the velocity crossed with the momentum on the right hand side because it is zero since the velocity is always parallel to the momentum.

$$\vec{\tau} = \vec{v} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt}$$

Using the definition of velocity and the product rule of derivatives,

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \frac{d}{dt}(\vec{r} \times \vec{p}).$$

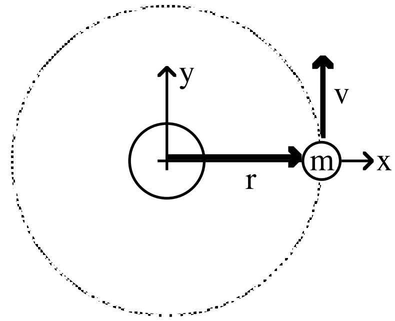
Equating the differentiated quantities,

$$\vec{L} = \vec{r} \times \vec{p}$$

We will make this the definition of angular momentum,

The Definition of Angular Momentum $\vec{L} \equiv \vec{r} \times \vec{p}$

Example 29.2: Find the angular momentum of the moon about Earth.



Given: $m = 7.36\text{x}10^{22}\,\text{kg}$, $r = 3.84\text{x}10^{8}\,\text{m}$ and
$T = 27.3\text{d} = 2.36\text{x}10^{6}\,\text{s}$.
Find: $\vec{L} = ?$

Using the definitions of speed and momentum,

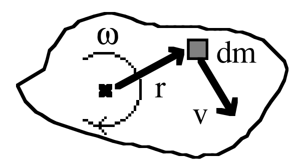
$$p \equiv mv = m\frac{2\pi r}{T} = (7.36\text{x}10^{22})\frac{2\pi(3.84\text{x}10^{8})}{2.36\text{x}10^{6}} = 7.52\text{x}10^{25}\,\text{kg}\cdot\text{m/s}$$

Using the definition of angular momentum and the fact that the radius and velocity are perpendicular,

$$\vec{L} \equiv \vec{r}\times\vec{p} \Rightarrow L = rp = (3.84\text{x}10^{8})(7.52\text{x}10^{25}) \Rightarrow \boxed{L = 2.89\text{x}10^{34}\,\text{kg}\cdot\text{m}^2/\text{s}}.$$

The direction is out of the page by the right hand rule.

This definition of angular momentum is the angular momentum of a particle that is small with respect to the distance from the center of rotation. That's why it worked in the last example for the moon. The moon is small compared to the distance between Earth and the moon.



A rigid body is generally about the same size as its distance from the rotation center. Let's check the definition of angular momentum to be sure it is consistent with the equation for the angular momentum of a rigid body. At the right is a rigid body spinning about the point shown. Applying the definitions of angular momentum and linear momentum to a small piece of the object dm gives,

$$\vec{L} \equiv \vec{r}\times\vec{p} \Rightarrow d\vec{L} = \vec{r}\times\vec{v}dm.$$

Since the pivot is fixed, the radius and velocity are always perpendicular and the angular momentum points into the page. We'll use $\hat{n}$ to indicate the direction of the angular velocity vector. Using the angular/linear rule to express the tangential velocity in terms of the angular speed,

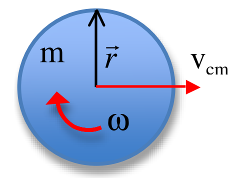
$$d\vec{L} = rvdm\hat{n} = r(r\omega)\hat{n}dm = r^2\omega\hat{n}dm = \vec{\omega}r^2dm.$$

Integrating over the entire object, keeping in mind that the angular velocity is the same for all dm's,

$$\int d\vec{L} = \int \vec{\omega}r^2dm \Rightarrow \vec{L} = \vec{\omega}\int r^2dm \Rightarrow \vec{L} = I\vec{\omega}.$$

So we can use $\vec{L} \equiv \vec{r}\times\vec{p}$ for point-like objects and $\vec{L} = I\vec{\omega}$ for extended objects.

Example 29.3: A bowling ball has a mass of 5.00kg and a radius of 11.0cm. Find the angular momentum if it rolls without slipping down the alley at 8.00m/s.



Given: m = 5.00kg, r = 0.110m, and v_{cm} = 8.00m/s.
Find: $\vec{L} = ?$

The ball is an extended object in this situation. It has a rotational inertia of $I = \frac{2}{5}mr^2$.
The angular momentum is then,

$$\vec{L} = I\vec{\omega} \Rightarrow L = \frac{2}{5}mr^2\omega.$$

Since it rolls without slipping,

$$v_{cm} = r\omega \Rightarrow \omega = \frac{v_{cm}}{r} \Rightarrow L = \frac{2}{5}mr^2\frac{v_{cm}}{r} = \frac{2}{5}mrv_{cm}.$$

Plugging in the values,

$L = \frac{2}{5}(5)(0.11)(8) \Rightarrow \boxed{L = 1.76\frac{kg\cdot m^2}{s}}$. The direction is out of the page by the right hand rule.

Section Summary

What do objects do and why do they do it? We have now added the idea of angular momentum to our understanding of motion. We followed the same path that led us to rewrite the Second Law in terms of linear momentum to rewrite the Second Law for Rotation,

$$\text{Newton's Second Law for Rotation } \Sigma\vec{\tau} = \frac{d\vec{L}}{dt}.$$

in terms of a new concept,

$$\text{The Definition of Angular Momentum } \vec{L} \equiv \vec{r} \times \vec{p}.$$

For a rigid object, the angular momentum can be calculated using,

$$\text{The Angular Momentum of a Rigid Body } \vec{L} = I\vec{\omega}.$$