Section 30 – Conservation of Angular Momentum

What do objects do and why do they do it? We can describe this motion with rotational kinematics and explain it in terms of torques and rotational kinetic energy. Now that we have determined torque as the cause of changes in angular momentum, we will use the analogy to the Law of Conservation of Linear Momentum, to introduce a new law, the Conservation of Angular Momentum.

Section Outline

1. A Review of Conservation of Linear Momentum
2. The Law of Conservation of Angular Momentum
3. Conservation of Angular Momentum in the Solar System

1. A Review of Conservation of Linear Momentum

The Original Second Law tells us the net external force on a system of objects causes the linear momentum to change,

$$\Sigma\vec{F} = \frac{d\vec{p}}{dt}\ .$$

When there are no external forces,

$$\Sigma\vec{F} = \frac{d\vec{p}}{dt} \Rightarrow 0 = \frac{d\vec{p}}{dt} \Rightarrow \Delta\vec{p} = 0\ .$$

In other words, the linear momentum is a constant. This is the Law of Conservation of Linear Momentum,

Law of Conservation of Linear Momentum

"The total linear momentum of an isolated systems of objects remains constant."

2. The Law of Conservation of Angular Momentum

When we can write the Second Law for Rotation in terms of the change in angular momentum,

$$\Sigma\vec{\tau} = \frac{d\vec{L}}{dt}\ ,$$

we can follow the same path as we did for linear momentum. Applying the Second Law for Rotation to a system with no external torques,

$$\Sigma\vec{\tau} = \frac{d\vec{L}}{dt} \Rightarrow 0 = \frac{d\vec{L}}{dt} \Rightarrow \Delta\vec{L} = 0\ .$$

So, the angular momentum will remain fixed and we have established the,

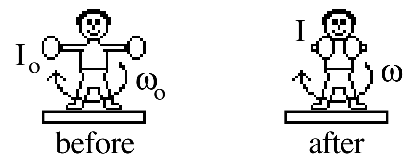
The Law of Conservation of Angular Momentum

"The total angular momentum of an isolated system of bodies remains constant."

The Law of Conservation of Angular Momentum explains why a top stays up as long as it is spinning. When you spin the top, you give it angular momentum which points either up or down depending on which way you spin it. As long as friction and other torques can be ignored, the angular momentum vector must stay constant. So, the top must stay vertical in order to keep the angular momentum vector pointed vertically.

Example 30.1: As an ice skater spins, he pulls his arms in and his spin rate changes from 10.0rpm to 15.0rpm. Find the factor by which he changed his (a)rotational inertia and (b)kinetic energy. (c)Explain the source of the increased kinetic energy.

Given: $\omega_o = 10.0\text{rpm}$ and $\omega = 15.0\text{rpm}$
Find: $\frac{I}{I_o} = ?$ and $\frac{K}{K_o} = ?$



(a)Using the angular momentum of a rigid body, the initial angular momentum is $L_o = I_o\omega_o$ and the final angular momentum is $L = I\omega$. Using the Law of Conservation of Angular Momentum,

$$L_o = L \Rightarrow I_o\omega_o = I\omega \Rightarrow \frac{I}{I_o} = \frac{\omega_o}{\omega} = \frac{10}{15} \Rightarrow \boxed{\frac{I}{I_o} = \frac{2}{3}}.$$

The rotation rate increases because the rotational inertia decreases.
(b)Using the rotational kinetic energy and the angular momentum of a rigid body,

$$\frac{K}{K_o} = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}I_o\omega_o^2} = \frac{\frac{L^2}{2I}}{\frac{L_o^2}{2I_o}} = \frac{L^2 I_o}{L_o^2 I}.$$

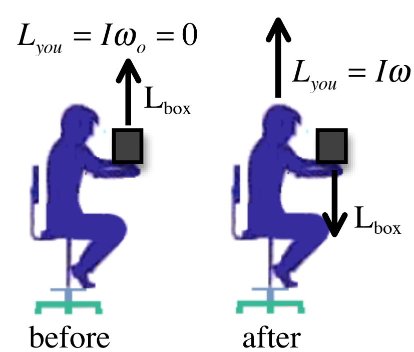
The Law of Conservation of Angular Momentum requires that $L_o = L$, so,

$$\frac{K}{K_o} = \frac{I_o}{I} \Rightarrow \boxed{\frac{K}{K_o} = \frac{3}{2}}.$$

(c)This increase in kinetic energy comes from the non-conservative forces exerted by the skater to pull his arms in. This energy ultimately comes from the food he eats.

We need to remember that angular momentum is a vector quantity.

Example 30.2: You sit on a stool at rest when a friend hands you a light-weight black box and asks you to flip the box over. When you do, you begin to spin on the stool at a rate of 8.00rpm counter-clockwise. Assume you are a 70.0kg cylinder of radius 13.0cm. Find the original angular momentum in the black box.



Given: $\omega_o = 0$, $\omega = 8.00\text{rpm} = 0.837\text{rad/s}$, $m = 70.0\text{kg}$, and $r = 0.130\text{m}$.
Find: $L_{box} = ?$

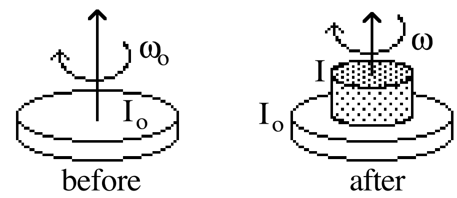
Let's start by estimating your rotational inertia, $I \approx \frac{1}{2}mr^2$.
The initial angular momentum of the box must be upward (counter-clockwise) because that is the direction you wind up spinning. The initial angular momentum of you and the box is just due to the box, $L_o = L_{box} + L_{you} = L_{box}$.
The final angular momentum of the system is, $L = L_{you} - L_{box} = I\omega - L_{box} = \frac{1}{2}mr^2\omega - L_{box}$.
Applying the Law of Conservation of Angular Momentum,

$$L = L_o \Rightarrow \tfrac{1}{2}mr^2\omega - L_{box} = L_{box} \Rightarrow \tfrac{1}{2}mr^2\omega = 2L_{box} \Rightarrow L_{box} = \tfrac{1}{4}mr^2\omega.$$

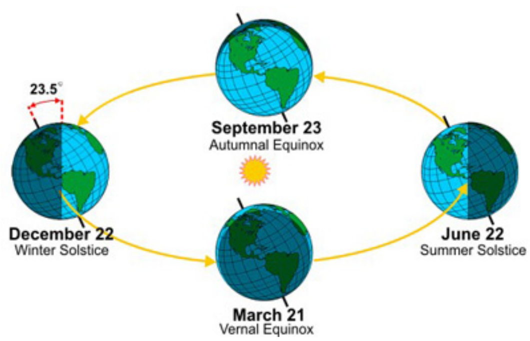
Plugging in the values,

$$L_{box} = \tfrac{1}{4}mr^2\omega = \tfrac{1}{4}(70)(0.13)^2(0.837) \Rightarrow L_{box} = 0.248\,\tfrac{kg\cdot m^2}{s}.$$

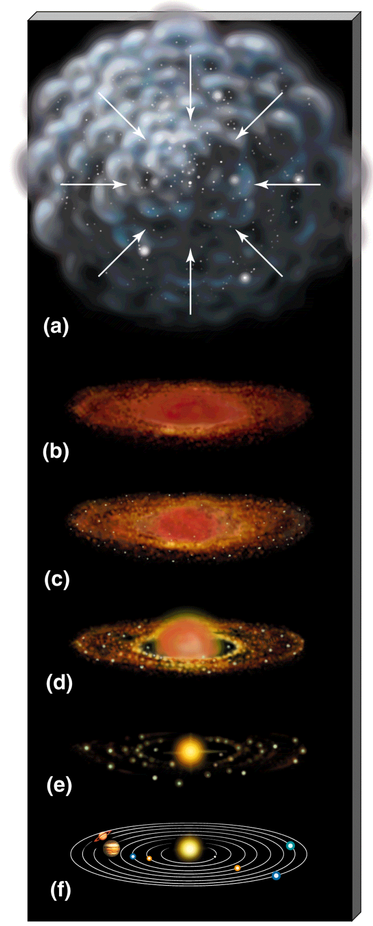
Just as linear momentum was useful to study collisions, so is angular momentum.

> *Example 30.3: A potter's wheel with a rotational inertia of 0.400kg·m² rotates at 100rpm when a 4.00kg cylindrical hunk of clay 20.0cm in diameter is thrown directly down on the center. Find the rotation rate just after this collision.*
>
> 
>
>
> Given: $I = 0.400\text{kg}\cdot\text{m}^2$, $\omega_o = 100\text{rpm}$, $m = 4.00\text{kg}$, and $r = 0.200\text{m}$.
> Find: $\omega = ?$
>
> Using the angular momentum of a rigid body, the initial angular momentum is,
> $$L_o = I_o\omega_o$$
> and the final angular momentum is,
> $$L = (I_o + I)\omega\,.$$
> Using the rotational inertia of a cylinder,
> $$L = \left(I_o + \tfrac{1}{2}mr^2\right)\omega\,.$$
> The initial and final angular momentum must be equal according to the Law of Conservation of Angular Momentum,
> $$I_o\omega_o = \left(I_o + \tfrac{1}{2}mr^2\right)\omega \Rightarrow \omega = \frac{I_o}{I_o + \frac{1}{2}mr^2}\omega_o = \frac{0.400}{0.400 + \frac{1}{2}(4.00)(0.100)^2}(100) \Rightarrow \boxed{\omega = 95.2\,\text{rpm}}.$$

3. <u>Conservation of Angular Momentum in the Solar System</u>



At the left is a sketch of Earth as it orbits the sun. The Earth's rotation axis is 23.5° from the perpendicular associated with the plane of its orbit. The angular momentum of Earth points along this axis regardless of where it is in its orbit. Law of Conservation of Angular Momentum requires it. So, whether it is December or June, the axis remains pointed the same direction in space. In December then, the direct rays of the sun will hit the southern hemisphere while in June, they will hit the northern hemisphere. In December it is summer in South America and winter in North America and the reverse is true in June. The Law of Conservation of Angular Momentum explains the seasons on Earth.



Another example of the Law of Conservation of Angular Momentum is the fact that all planets in our solar system orbit in the same plane and in the same direction. At the right is a drawing of the formation of the solar system. Each image will be described below:

a. A large cloud of interstellar gas begins to coalesce under the attraction of gravitational forces. If we added up all the angular momentum of all the particles, the result would point along some direction because the chances of it being exactly zero are very small. Let's take the direction of the angular momentum to be upward.
b. As the gas gets pulled together by gravity, it can easily contract along the angular momentum axis without changing the angular momentum. However, in the plane perpendicular to the axis it struggles to collapse. If the gas comes inward toward the spin axis, then the angular velocity would increase just like the ice skater when pulling in their arms. Since there is little energy available to do this, it cannot happen. So, the Law of Conservation of Angular Momentum explains why everything is forced into a disk spinning.
c. Gas continues to coalesce and the plane gets flatter. Note that there must be more gas in the center than near the edges due to the flattening of the originally spherical cloud.
d. The gas in the middle finally reaches the temperature and density to ignite a central star – the sun.
e. The remaining material coalesces into planets.
f. Eventually, everything that is not in a spherical orbit rotating the correct direction collides with something that does orbits normally. All the remaining planets are in one plane, orbiting the same direction as each other, and the same direction as the sun spins.

So, next time you spin a top remember that the reason is doesn't fall over while it spins is the same reason the Solar System is like it is!

<u>Section Summary</u>

What do objects do and why do they do it? We followed the same path that led us from the Second Law to the Law of Conservation of Linear Momentum to go from the Second Law for Rotation to establish our third conservation law,

The Law of Conservation of Angular Momentum
"The total angular momentum of an isolated system of bodies remains constant."

The Law of Conservation of Angular Momentum explains many rotating systems from why a spinning top stays upright to the seasons of the year to the properties of the Solar System.