Section 32 – Static Equilibrium

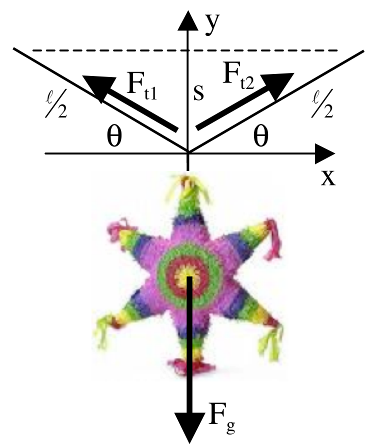
Now we have a fairly complete answer to our question, "what do objects do and why do they do it?" We will spend the rest of the semester applying our knowledge to a variety of new situations. In this section, we will focus our attention on objects that are at rest and remain at rest. Such objects are said to be in "static equilibrium." This is highly desired for buildings, bridges and many other structures.

An object in static equilibrium has no acceleration and no angular acceleration. According to the Second Laws for Translation and Rotation,

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \Sigma\vec{F} = 0 \text{ and } \Sigma\vec{\tau} = m\vec{\alpha} \Rightarrow \Sigma\vec{\tau} = 0.$$

That is, the net force on the object and the net torque on the object must be zero. The rest of this section consists of example problems of increasing complexity.

Example 32.1: A 5.00kg piñata is hung from the middle of a 4.00m long rope. Find the tension in the rope if it sags 20.0cm.



Given: m = 5.00kg, ℓ = 4.00m, and s = 20.0cm
Find: F_{t1} = ? and F_{t2} = ?

The angle that the tension makes with the horizontal can be found using the definition of the sine,

$$\sin\theta = \frac{s}{\frac{\ell}{2}} = \frac{2s}{\ell} \Rightarrow \theta = \arcsin\left(\frac{2s}{\ell}\right) = \arcsin\left(\frac{2(0.2)}{4}\right) \Rightarrow \theta = 5.74^\circ$$

Applying the Second Law to the point where the rope connects to the piñata,

$$\Sigma F_x = ma_x \Rightarrow F_{t2}\cos\theta - F_{t1}\cos\theta = 0 \Rightarrow F_{t2} = F_{t1} = F_t$$

Since the piñata is in the middle of the rope, both angles are equal and the tension in both sides must be the same. Looking at the y-direction,

$$\Sigma F_y = ma_y \Rightarrow F_t\sin\theta + F_t\sin\theta - mg = 0 \Rightarrow 2F_t\sin\theta = mg \Rightarrow F_t = \frac{mg}{2\sin\theta}$$

Putting in the numbers,

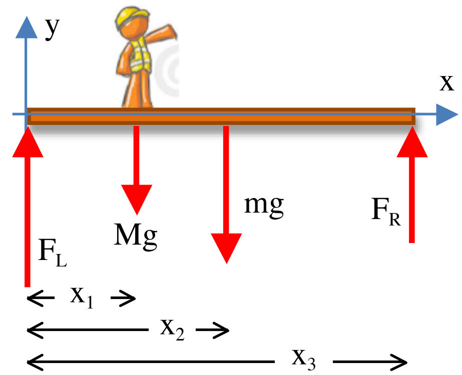
$$F_t = \frac{(5.00)(9.80)}{2\sin 5.74^\circ} \Rightarrow \boxed{F_t = 245\text{N}}.$$

This problem did not require the use of torques.

Now, let's look at a problem that involves some torques.

Example 32.2: A 20.0kg board 4.00m long is supported at each end. A 70.0kg person stands 1.00m from the left side. Find the force exerted by each support.

Given: m = 20.0kg, M = 70.0kg, x_1 = 1.00m, x_2 = 2.00m, and x_3 = 4.00m
Find: F_L = ? and F_R = ?



Note that choosing the origin where one of the unknown forces acts simplifies the algebra of the solution because the torque about the origin caused by this unknown force will be zero. So this unknown won't appear in the equation produced by the Second Law of Rotation.

Applying the Second Law for Rotation to the board about the origin using the convention that counterclockwise torques are positive,

$$\Sigma\tau_o = I\alpha \Rightarrow (0)F_L - x_1 Mg - x_2 mg + x_3 F_R = 0 .$$

Choosing the origin as we did means that this equation depends on only one unknown, F_R, so we can solve directly,

$$F_R = \frac{x_1 Mg + x_2 mg}{x_3} \Rightarrow F_R = \frac{(1)(70)(9.8) + (2)(20)(9.8)}{4.00} \Rightarrow \boxed{F_R = 270\text{N}}.$$

Applying the Second Law for Translation to the board,

$$\Sigma F_y = ma_y \Rightarrow F_L + F_R - Mg - mg = 0 \Rightarrow F_L = Mg + mg - F_R .$$

Putting in the numbers,

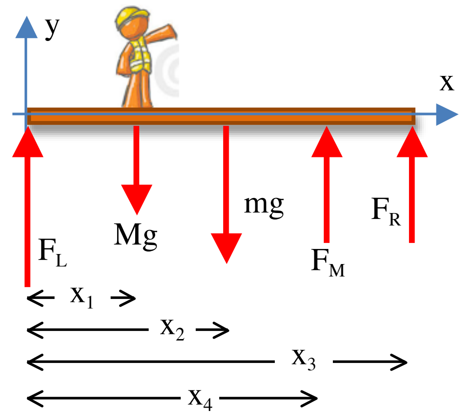
$$F_L = (70)(9.8) + (20)(9.8) - 270 \Rightarrow \boxed{F_L = 612N}.$$

The result would be the same if we had chosen to take the torques about another point such as the center-of-mass. Then the Second Law for Rotation would require,

$$\Sigma\tau_{cm} = I\alpha \Rightarrow -x_2 F_L + (x_2 - x_1)Mg + (0)mg + (x_3 - x_2)F_R = 0 .$$

If we used this equation with the equation from the Second Law for Translation and solve the system of two equations and two unknowns, we would get the same numerical values for F_1 and F_2. The math just would have been harder. So, the good news is that you can't get the wrong answer by making the "wrong" choice about where to take the torques. At worst you'll have harder algebra.

Example 32.3: A 20.0kg board 4.00m long is supported at each end. A 70.0kg person stands 1.00m from the left side. An additional support is located 3.00m from the left side. Find the force exerted by each support.



Given: m = 20.0kg, M = 70.0kg, x_1 = 1.00m, x_2 = 2.00m, and x_3 = 4.00m
Find: F_L = ? and F_R = ?

Adding an additional support to the last example should be straightforward.

Applying the Second Law for Rotation to the board about the origin using the convention that counterclockwise torques are positive,

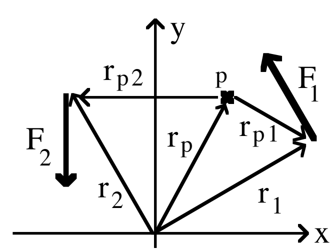
$$\Sigma\tau_o = I\alpha \Rightarrow (0)F_L - x_1 Mg - x_2 mg + x_3 F_R + x_4 F_M = 0 \; .$$

Applying the Second Law for Translation to the board,

$$\Sigma F_y = ma_y \Rightarrow F_L + F_R + F_M - Mg - mg = 0 \; .$$

Perhaps you can see the problem now. We have two equations and three unknowns. So, there is no unique answer to this problem. This is called a "statically indeterminate" situation. The lesson here is that there is no point in pursuing the numerical solution to a problem unless you have as many equations as unknowns.

Since torque depends on the choice of pivot point, the question is, about which points is the torque zero? Below is an object in equilibrium acted upon by a collection of forces (only two are shown). The total torque about the origin is known to be zero because the object stays at rest. It will be shown that the total torque about any point, P, is also zero.



The torque about P is given by,

$$\Sigma\vec{\tau}_P = \vec{r}_{P1} \times \vec{F}_1 + \vec{r}_{P2} \times \vec{F}_2 + \cdots \; .$$

The position vectors can be rewritten as,

$$\vec{r}_P + \vec{r}_{Pi} = \vec{r}_i \Rightarrow \vec{r}_{Pi} = \vec{r}_i - \vec{r}_P \; .$$

The torque about P is now,

$$\Sigma\vec{\tau}_P = \left(\vec{r}_1 - \vec{r}_P\right) \times \vec{F}_1 + \left(\vec{r}_2 - \vec{r}_P\right) \times \vec{F}_2 + \cdots \; .$$

Using the distributive property,

$$\Sigma\vec{\tau}_P = \vec{r}_1 \times \vec{F}_1 - \vec{r}_P \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 - \vec{r}_P \times \vec{F}_2 + \cdots \; .$$

Rearranging the terms,

$$\Sigma\vec{\tau}_P = \left[\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots\right] - \left[\vec{r}_P \times \vec{F}_1 + \vec{r}_P \times \vec{F}_2 + \cdots\right] = \left[\vec{r}_1 \times \vec{F}_1 + \vec{r}_2 \times \vec{F}_2 + \cdots\right] - \vec{r}_P \times \left[\vec{F}_1 + \vec{F}_2 + \cdots\right].$$

The first term is the torque about the origin and the brackets in the second term enclose the total force,

$$\Sigma\vec{\tau}_P = \Sigma\vec{\tau}_o - \vec{r}_P \times \Sigma\vec{F} \; .$$

Since the torque about the origin is zero and the net force on the object is zero, the torque about any point will be zero for an object in equilibrium,

$$\Sigma\vec{\tau}_P = 0 \; .$$

The point of all this is that we are free to pick the point about which we take the torque knowing that the sum of the torques will be zero regardless. We can take advantage of this, as we did in the last problem, to simplify the mathematics required to get the answer.

COMMENT ON PROBLEM SOLVING:
Before applying the Second Law for Rotation you must choose a point about which to calculate the torques. You must indicate this point in your solution, either in the picture or explicitly in the explanation. It is usually the best policy to choose this point to be a place where unknown forces act. Generally, this will reduce the difficulty of the mathematics needed to get the numerical answers.

Section Summary

We didn't develop any new ideas of physics in the section. However, we did learn to apply our knowledge of what objects do and why they do it to systems in static equilibrium. That is, in situations where $\Sigma\vec{F} = 0$ and $\Sigma\vec{\tau}_P = 0$. We found that if a system is in static equilibrium, then the torque about any point must be zero. We also learned that a unique numerical solution is only possible when the equilibrium conditions provide as many equations as there are unknowns. When there are to few equations for the number of unknowns, the problem is statically indeterminate.