Section 34 – Hooke's Rule and Simple Harmonic Motion

The goal of this course is to answer the questions, "What do objects do?" and "Why do they do it?" In the next three sections we investigate a new form of motion called "Simple Harmonic Motion" or SHM. A typical example is a mass vibrating at the end of a spring. To answer the questions, we will use the tools we've developed over the course to date. In this section, we will investigate the properties of the force involved. Then we will apply the Second Law to derive the equations that describe the motion. These equations predict the position as a function of time x(t), the velocity as a function of time v(t), the acceleration as a function of time a(t), the velocity as a function of position v(x), and the acceleration as a function of position a(x).

Section Outline

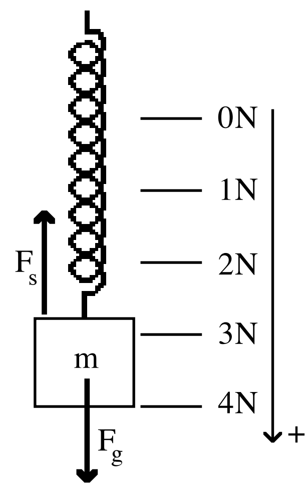
1. The Spring Force - Hooke's Rule
2. The Description of Simple Harmonic Motion

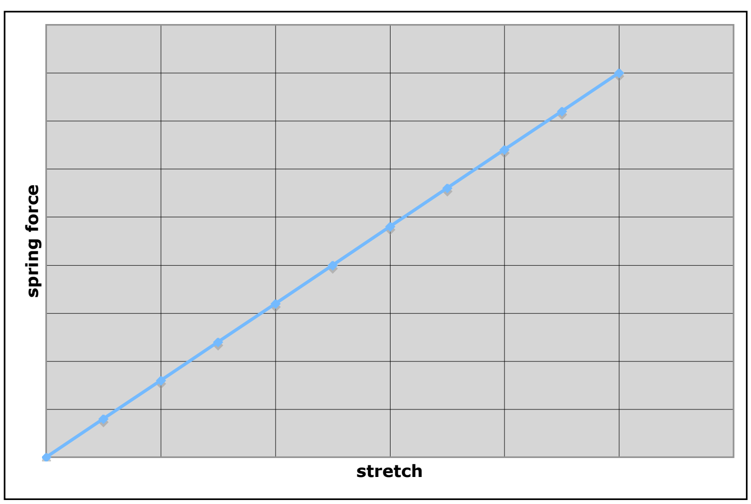
1. The Spring Force - Hooke's Rule

When a masses is in equilibrium hanging from a spring, the Second Law requires that,

$$\Sigma F = ma \Rightarrow F_s - F_g = 0 \Rightarrow F_s = F_g .$$

The sketch at the right shows the effect of adding 1N weights one at a time to the spring. Each additional weight stretches the spring the same amount. The graph of the force exerted by the spring versus the total weight is shown below. It is a straight line.





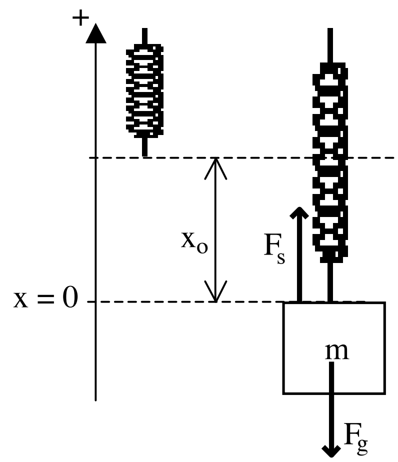
The graph shows that the force due to the spring is proportional to the stretch of the spring. Using F_s for the spring force, x for the stretch of the spring, and k for the slope of the graph, the equation for a straight line through the origin tells us that,

$$F_s = kx .$$

The slope of the graph is called the "spring constant." Since the stretch is in the opposite direction from the force exerted by the spring, we need a negative sign to write a vector equation. This equation is called,

$$\text{Hooke's Rule } \vec{F}_s = -k\vec{x}$$

Example 34.1: Find the spring constant of a spring that stretches 50.0cm when 500g is hung from it.



Given: $x_o = 0.500\text{m}$ and $m = 0.500\text{kg}$
Find: k = ?

Applying the Second Law to the mass at rest at the end of the spring,

$$\Sigma F = ma \Rightarrow F_s - F_g = 0 \Rightarrow F_s = F_g .$$

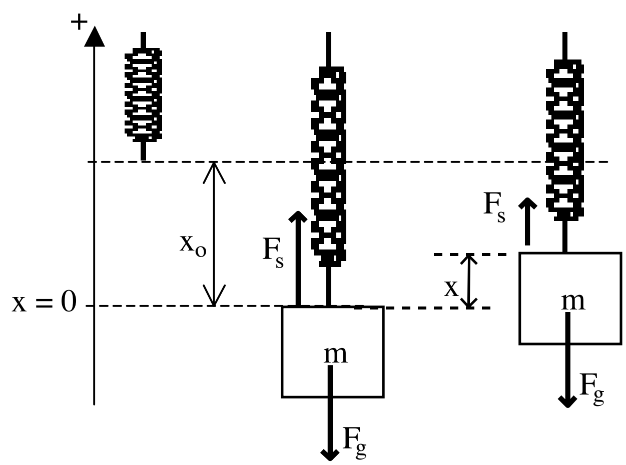
Using Hooke's Rule and the mass/weight rule,

$$F_s = F_g \Rightarrow kx_o = mg ,$$

where x_o is the stretch of the spring. Solving for the spring constant and plugging in estimates for the values,

$$k = \frac{mg}{x_o} = \frac{(0.500)(9.80)}{(0.500)} \Rightarrow \boxed{k = 9.80\,\text{N/m}}.$$

2. The Description of Simple Harmonic Motion



The general procedure for finding the motion of an object is to apply the Second Law to get the acceleration. Then use the definitions of acceleration and velocity to get the position as a function of time x(t), the velocity as a function of time v(t), the acceleration as a function of time a(t), the velocity as a function of position v(x), and the acceleration as a function of position a(x).

We want to study oscillatory motion, not equilibrium. So we'll, lift the mass a distance, A, above the equilibrium position, x_o, and release it. Some time later, it is at the position, x, above the equilibrium. Applying the Second Law to the mass at this point,

$$\Sigma F = ma \Rightarrow F_s - F_g = ma .$$

Notice the coordinates have upward as positive and the origin at the equilibrium position. The stretch at the point we are looking at is x_o-x, so using Hooke's Rule,

$$k(x_o - x) - mg = ma .$$

Applying the Second Law to the equilibrium situation,

$$\Sigma F = ma \Rightarrow F_s - F_g = 0 \Rightarrow kx_o = mg .$$

Substituting into the non-equilibrium equation,

$$k(x_o - x) - kx_o = ma \Rightarrow -kx = ma \Rightarrow a = -\frac{k}{m}x .$$

This is the acceleration as a function of position. For convenience we define,

$$\omega^2 \equiv \frac{k}{m} .$$

Now we can write,

$$a(x) = -\omega^2 x .$$

Now use the definitions of acceleration and velocity, as well as, the chain rule from calculus,

$$a \equiv \frac{dv}{dt} = \frac{dx}{dt} \cdot \frac{dv}{dx} \Rightarrow v\frac{dv}{dx} = -\omega^2 x \,.$$

Cross multiplying and integrating,

$$vdv = -\omega^2 xdx \Rightarrow \int_0^v vdv = -\omega^2 \int_A^x xdx \,.$$

The limits are determined because the velocity is zero when the mass is released from x = A. A is called the "amplitude." Completing the integration,

$$\int_0^v vdv = -\omega^2 \int_A^x xdx \Rightarrow \tfrac{1}{2}v^2 = -\omega^2\left(\tfrac{1}{2}x^2 - \tfrac{1}{2}A^2\right) \Rightarrow v^2 = \omega^2\left(A^2 - x^2\right).$$

The velocity as a function of position is,

$$v(x) = \pm\omega\sqrt{A^2 - x^2} \,.$$

The position as a function of time can be found by using the definition of velocity and integrating once more,

$$v \equiv \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \omega\sqrt{A^2 - x^2} \Rightarrow \frac{dx}{\sqrt{A^2 - x^2}} = \omega dt \Rightarrow \int_A^x \frac{dx}{\sqrt{A^2 - x^2}} = \int_0^t \omega\, dt \,.$$

In this case, x_o = A, but in general it could be any value less than or equal to A. Completing the integration,

$$\int_{x_o}^x \frac{dx}{\sqrt{A^2 - x^2}} = \int_0^t \omega\, dt \Rightarrow \left[-\arccos\frac{x}{A}\right]_A^x = \omega t \Rightarrow -\arccos\frac{x}{A} + \arccos\frac{A}{A} = \omega t \,.$$

Since the arcos(1) is zero,

$$x(t) = A\cos\omega t \,.$$

Since we can't always be sure that the mass is going to be raised up and released, we have to allow for other ways for the mass to begin oscillating such as being kicked downward from equilibrium. The standard way to do this is to introduce the "phase angle" which we will discuss in more detail later. Finally, the position as a function of time is,

$$x(t) = A\cos(\omega t + \delta) \,.$$

The velocity as a function of time can be found by using the definition of velocity,

$$v \equiv \frac{dx}{dt} \Rightarrow v = \frac{d}{dt}A\cos(\omega t + \delta) \,.$$

The result is,

$$v(t) = -\omega A\sin(\omega t + \delta) \,.$$

Using the definition of acceleration and differentiating again,

$$a \equiv \frac{dv}{dt} \Rightarrow a = \frac{d}{dt}\left[-\omega A\sin(\omega t + \delta)\right].$$

The result is,

$$a(t) = -\omega^2 A\cos(\omega t + \delta) \,.$$

In summary, we have started with the Second Law, using the forces on the mass at the end of the spring to get acceleration. From the acceleration and the definitions of acceleration and velocity we have found the equations of motion. That is, we have position as a function of time x(t), the velocity as a function of time v(t), the acceleration as a function of time a(t), the velocity as a function of position v(x), and the acceleration as a function of position a(x). These are the equations of motion for simple harmonic motion.

SHM Equations of Motion

$x(t) = A\cos(\omega t + \delta)$ $\quad v(t) = -\omega A\sin(\omega t + \delta)$ $\quad a(t) = -\omega^2 A\cos(\omega t + \delta)$

$v(x) = \pm\omega\sqrt{A^2 - x^2}$ $\quad a(x) = -\omega^2 x$,

where A is the amplitude, δ is the phase angle, and the angular frequency is $\omega = \sqrt{\frac{k}{m}}$.

The sines & cosines mean harmonic or oscillatory motion. The word "simple" is used because these are the most basic harmonic functions (there are more complicated ones). Notice that the sines and cosines come about because the acceleration is equal to minus a constant times the position. The only functions that produce the negative of themselves after two derivatives are sine and cosine. Therefore, whenever a system has an acceleration that is proportional to the negative of the position SHM is the result.

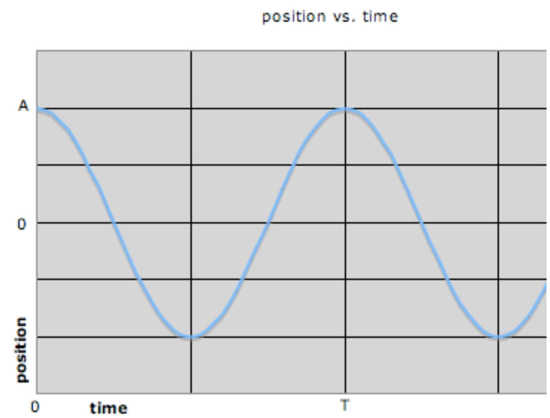
At the right is a graph of position versus time for the mass released from rest at x = A. The equation for this graph we just found to be,

$$x(t) = A\cos\omega t.$$

Notice that x = A at t = 0 and t =T. So,

$$A = A\cos 0 = A\cos\omega T \Rightarrow 1 = \cos\omega T \Rightarrow \omega T = 2\pi \Rightarrow \omega = \tfrac{2\pi}{T}$$

In other words, ω is an angular frequency.



Angular Frequency of a Spring $\omega = \sqrt{\frac{k}{m}}$

The SHM equations of motion are the equivalent of the kinematics equations, the equations of motion for constant acceleration. However, the SHM equations involve trigonometric functions, so we'll need to practice a bit to get familiar with there use.

Example 34.2: The mass in the last example is raised 10.0cm and released from rest. Find (a)the angular frequency, (b)the amplitude and (c)the phase angle.

Given: k = 9.80N/m, m = 0.500kg, A = 10.0cm, and x(0) = 10.0cm.
Find: ω = ?, A = ?, and δ = ?

(a)The angular frequency for a spring is,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{9.80}{0.500}} \Rightarrow \boxed{\omega = 4.43\text{rad/s}}.$$

(b)The amplitude is the maximum displacement given to be $\boxed{A = 10.0\text{cm}}$.
(c)We are told that x = A at t = 0. Using the equation for position as a function of time,

$$x(t) = A\cos(\omega t + \delta) \Rightarrow A = A\cos\delta \Rightarrow \cos\delta = 1 \Rightarrow \boxed{\delta = 0}.$$

Example 34.3: Find the time it takes to get back to the equilibrium point and the velocity when it gets there.

Given: $\omega = 4.43\text{rad/s}$, $A = 0.100\text{m}$, and $\delta = 0$.
Find: $t = ?$ and $v = ?$ at $x = 0$.

Since we want a time at a given position ($x = 0$), we should use the position as a function of time,

$$x(t) = A\cos(\omega t + \delta) \Rightarrow 0 = \cos(\omega t + \delta) \Rightarrow \omega t + \delta = \tfrac{\pi}{2}.$$

From example 2 we know the phase angle is zero,

$$\omega t = \frac{\pi}{2} \Rightarrow t = \frac{\pi}{2\omega} = \frac{\pi}{2(4.43)} \Rightarrow \boxed{t = 0.355\text{s}}.$$

Using the velocity as a function of time,

$$v(t) = -\omega A\sin(\omega t + \delta) \Rightarrow v = -\omega A\sin\left(\tfrac{\pi}{2} + 0\right) = -\omega A = -(4.43)(0.100) \Rightarrow \boxed{v = -0.443\text{m/s}}.$$

The minus means downward.

Example 34.4: Find expressions for (a)the maximum speed, (b)the maximum acceleration and (c)the period of SHM in terms of the angular frequency and amplitude.

Given: ω, A, and δ.
Find: $v_{max} = ?$, $a_{max} = ?$, and $T = ?$

(a)Using the velocity as a function of position,

$$v(x) = \pm\omega\sqrt{A^2 - x^2}.$$

By inspection you can see that the speed will be maximum when $x = 0$ (at equilibrium),

$$v(x) = \pm\omega\sqrt{A^2 - 0^2} \Rightarrow \boxed{v_{max} = \omega A}.$$

(b)By looking at the equation for acceleration as a function of position, the acceleration will be maximum when x is maximum ($x = A$),

$$a(x) = -\omega^2 x \Rightarrow \boxed{a_{max} = \omega^2 A}.$$

These last two results can also be obtained by looking at the velocity and acceleration as a function of time. These results also make sense in that the mass is moving fastest through the equilibrium and the acceleration is the largest at the extremes of the motion where the force exerted by the spring is the biggest.

(c)The period can be found from the position as a function of time. The position must be the same after one period so,

$$x(t) = A\cos(\omega t + \delta) = A\cos[\omega(t + T) + \delta] \Rightarrow \omega t + \delta + 2\pi = \omega(t + T) + \delta \Rightarrow 2\pi = \omega T \Rightarrow$$

$$\boxed{T = \frac{2\pi}{\omega} \Rightarrow \omega = 2\pi f},$$

just like with circular motion.

Section Summary

Why do objects do what they do? One answer is, they do what they do because of the forces that act on them. Since we are trying to gain a deeper understanding of oscillatory motion we began by examining the force exerted by a spring and established,

Hooke's Rule $\vec{F}_s = -k\vec{x}$.

The motion of a mass at the end of a spring was then described by applying the Second Law to get the acceleration $a = -\frac{k}{m}x$. Then we used the definitions of acceleration and velocity to get the

Equations of Motion for SHM:

$$\begin{gathered} a(x) = -\omega^2 x \\ v(x) = \pm\omega\sqrt{A^2 - x^2} \\ x(t) = A\cos(\omega t + \delta) \\ v(t) = -\omega A\sin(\omega t + \delta) \\ a(t) = -\omega^2 A\cos(\omega t + \delta) \end{gathered}$$

where A is the amplitude of the motion, ω is the angular frequency, and δ is the phase angle. The amplitude is the largest value of x, ω is related to the period of oscillation, and δ is determined by what fraction of an oscillation the system is at when it starts. For a mass on the end of a spring, $\omega = \sqrt{\frac{k}{m}}$

All of these equations follow from the first one. So any time we have a system where the acceleration is equal to the negative of a constant times the position, we will get this SHM and the angular frequency will be equal to the root of the constant.