Section 35 – SHM and Circular Motion

"What do objects do?" and "Why do they do it?" Objects sometimes oscillate in simple harmonic motion. In the last section we looked at mass vibrating at the end of a spring. We applied the Second Law to derive the equations of motion for SHM. In the process, we seemed to be using the idea of angular frequency just as we did when we looked at uniform circular motion. In addition, the equations of motion for SHM look every similar to the equations of motion for uniform circular motion. In this section, we will investigate the connection between SHM and uniform circular motion.

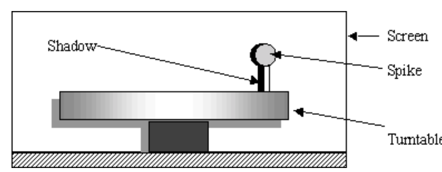
Next, we'll continue to build our understanding of SHM by looking at the oscillatory motion of a simple pendulum. We'll discover that it also is SHM under certain conditions.

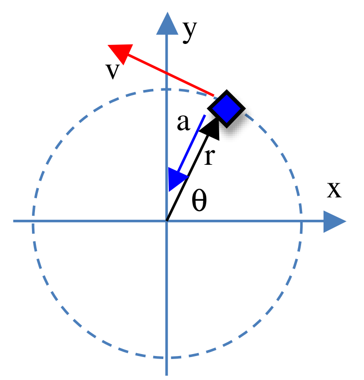
Section Outline

1. The Connection Between Uniform Circular Motion and SHM
2. Energy in SHM
3. The Simple Pendulum

1. The Connection Between Uniform Circular Motion and SHM

We have been using the idea of angular frequency as we did when we discussed circular motion. There must be some connection, so let's investigate. At the right is an object going in a circle on a rotating turntable. Just behind the object is a screen where the shadow of the object can be seen. The shadow moves back and forth as the object goes in a circle. The shadow appears to be in SHM.





At the right is a sketch of the object in uniform circular motion. It has a centripetal acceleration, a tangential velocity, and a position vector all shown. If we redraw the three vectors with their tails at the origin we can imagine all three spinning as the object rotates. Finding the x-components of each – the position, velocity, and acceleration of the shadow,

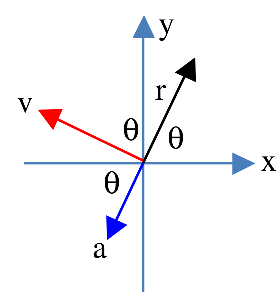
$$x = r\cos\theta,\; v_x = -v\sin\theta, \text{ and } a_x = -a\cos\theta.$$

The tangential velocity is related to this angular velocity,

$$v = \omega r.$$

Also, the centripetal acceleration is related to the angular velocity,

$$a = \frac{v^2}{r} = \frac{(\omega r)^2}{r} \Rightarrow a = \omega^2 r.$$



Substituting, we get,

$$x = r\cos\theta,\; v_x = -\omega r\sin\theta, \text{ and } a_x = -\omega^2 r\cos\theta.$$

Notice, just like SHM we have $a_x = -\omega^2 x$. The angle θ changes with time. We can write this using the definition of angular frequency,

$$\omega \equiv \frac{d\theta}{dt} \Rightarrow \int d\theta = \int \omega dt \Rightarrow \theta = \omega t + \delta.$$

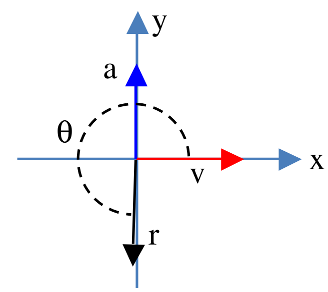
Now we see another way of looking at the phase angle, δ, as just an integration constant.

Finally, we can write the x-components of the, position, velocity and acceleration for the oscillating shadow as a function of time,

$$x(t) = r\cos(\omega t + \delta)\,,\ v(t) = -\omega r\sin(\omega t + \delta)\,,\text{ and }\ a(t) = -\omega^2 r\cos(\omega t + \delta)\,.$$

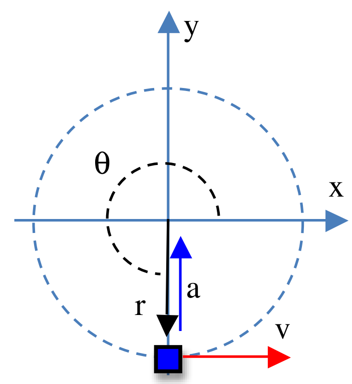
These are the same as the SHM equations of motion with A instead of r. The x component of the motion of an object in uniform circular motion is SHM. That explains why we keep talking about angular frequencies!

Example 35.1: A 500g mass rests in equilibrium at the end of a horizontal spring with spring constant 9.80N/m. The mass is given a sharp kick resulting in an initial velocity of 0.443m/s to the right. (a)Sketch the initial position, velocity, and acceleration vectors as if the object were in circular motion. Find (b)the location of the equivalent object in circular motion, (c)the phase angle, and (d)the equation for v(t).



Given: k = 9.80N/m, m = 0.500kg, v(0) = 0.443m/s, and x(0) = 0.
Find: $\vec{r} = ?$, $\vec{v} = ?$, $\vec{a} = ?$, d = ?, and x(t)=?

(a)We are given that the velocity vector is to the right and the initial position is zero. Therefore, the x-component of the velocity must be at a maximum and point to the right. The x-component of the position and acceleration vectors must be zero. The acceleration must point toward the center of the circle and the position must point outward. The answer then is in the sketch at the upper right.



(b)The equivalent object in circular motion with the vectors pointing the right direction must be as shown at the right.

(c)Looking at the circle, the phase angle must be $\boxed{270^\circ \text{ or } \frac{3\pi}{2}}$.

(d)Using the appropriate equation of motion for circular motion, the equation for the position as a function of time is

$$v(t) = -v_o \sin(\omega t + \delta)\,.$$

The angular frequency for a spring is,

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{9.8}{.5}} = 4.43\tfrac{rad}{s}\,.$$

So,

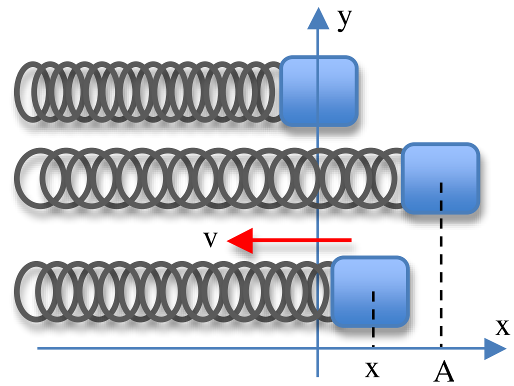
$$v(t) = -0.443\sin(4.43t + \tfrac{3\pi}{2})\,.$$

Note this results in v(0) = +0.443m/s as required.

2. Energy in SHM

We have already looked at energy in SHM. The result was that the potential energy stored in a spring was given by,

$$U_s = \tfrac{1}{2}kx^2 .$$



So, let's look at an oscillating mass at the end of a spring. In the top image at the right the mass is at rest at the equilibrium position of the spring. In the middle, image the mass has been pulled to the right a distance A. The system has a total energy equal to the potential energy in the spring,

$$E_o = U_s = \tfrac{1}{2}kA^2 .$$

The lower image is after the mass has been released and it is heading back to the left. It has a speed v when it reaches the position x. There is still some potential energy in the spring plus some kinetic energy,

$$E = K + U_s = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 .$$

Applying the Law of Conservation of Energy,

$$E_o = E \Rightarrow \tfrac{1}{2}kA^2 = \tfrac{1}{2}mv^2 + \tfrac{1}{2}kx^2 ,$$

and solving for the speed,

$$v = \pm\sqrt{\tfrac{k}{m}(A^2 - x^2)} .$$

Recall for a spring, $\omega = \sqrt{\tfrac{k}{m}}$, so the speed can be written as,

$$v = \pm\omega\sqrt{(A^2 - x^2)} .$$

This is the equation of motion for the speed as a function of position. Earlier we found this equation by applying the Second Law and the definitions of velocity and acceleration. Now, we see it is just an expression of the Law of Conservation of Energy.

3. The Simple Pendulum

A mass at the end of a string can certainly oscillate. The question is, is it SHM. Recall that the condition for SHM is,

$$a(x) = -\omega^2 x ,$$

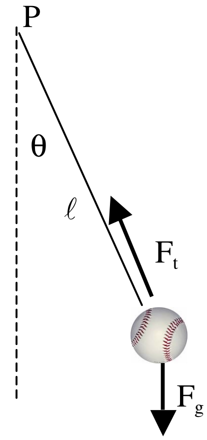
where ω is a constant. For a mass at the end of a spring, Newton's Second Law gave us,

$$a(x) = -\tfrac{k}{m}x .$$

From this equation we found deduced that the motion was SHM with an angular frequency equal to the root of the constants on the right hand side,

$$\omega = \sqrt{\tfrac{k}{m}} .$$

If we apply the Second Law to other systems and find that the acceleration is equal to the negative of some constants multiplied by the position, then we can follow the same logic to deduce that the motion will be SHM with an angular frequency equal to the root of the constants. Let's look at some oscillatory systems and see if they are, in fact, SHM.



A "simple" pendulum consists of a very light string with a concentrated mass at the end. The forces on the mass at the end are gravity and the tension. However, the tension exerts no torque about the top of the string. Applying the Second Law for Rotation,

$$\Sigma\tau_p = I\alpha \Rightarrow -mg\ell\sin\theta = m\ell^2\alpha \Rightarrow \alpha = -\frac{g}{\ell}\sin\theta .$$

Since we are looking at rotational motion, we are checking to see if the angular acceleration is equal to the negative of some constants multiplied by the angular position. Sadly, this is not the case for the simple pendulum because we have a $\sin\theta$ instead of just θ. However, for small angles,

$$\sin\theta \approx \theta \Rightarrow \alpha \approx -\frac{g}{\ell}\theta .$$

This is the SHM equation, the acceleration (angular, in this case) is equal to minus some constants times the position (again, angular). This is the same equation we got for the motion of the mass on the end of spring, except that θ replaces x. In other words, the equations of motion for the angle, θ, will be the simple harmonic motion equations with an angular frequency equal to the root of the constants so long as the angle is small.

Angular Frequency of a Simple Pendulum $\omega = \sqrt{\frac{g}{\ell}}$

Example 35.2: The pendulum in a grandfather clock must have a period of 2.00s so that each swing moves the second hand twice. Find the length of the pendulum.

Given: T = 2.00s
Find: $\ell = ?$

The angular frequency of a simple pendulum is,

$$\omega = \sqrt{\frac{g}{\ell}} .$$

It is related to the period,

$$\omega = 2\pi f = \frac{2\pi}{T} \Rightarrow T = \frac{2\pi}{\omega} \Rightarrow T = 2\pi\sqrt{\frac{\ell}{g}} \Rightarrow \ell = \frac{gT^2}{4\pi^2} .$$

Plugging in the numbers,

$$\ell = \frac{(9.80)(2.00)^2}{4\pi^2} \Rightarrow \boxed{\ell = 0.993m} .$$

This explains why all pendulums in grandfather clocks are about this size.

Section Summary

Why do objects do what they do? We have been building our understanding of simple harmonic motion. We have learned that an object with an acceleration that is equal to minus the product of some constant and the position is in SHM an obeys the SHM equations of motion,

$$
\begin{gathered}
a(x) = -\omega^2 x \\
v(x) = \pm\omega\sqrt{A^2 - x^2} \\
x(t) = A\cos(\omega t + \delta) \\
v(t) = -\omega A\sin(\omega t + \delta) \\
a(t) = -\omega^2 A\cos(\omega t + \delta)
\end{gathered}
$$

where A is the amplitude of the motion, ω is the angular frequency, and δ is the phase angle. The angular frequency will be equal to the root of the constants.

We examined the connection between circular motion and SHM. SHM is the motion of the shadow of an object in uniform circular motion. In other words, the equations of motion for the x-component of uniform circular motion are identical to the equations of motion for SHM.

With the knowledge above, we look at the oscillations of a simple pendulum and found that they are indeed SHM with an angular frequency given by,

$$\omega = \sqrt{\frac{g}{\ell}}\,.$$