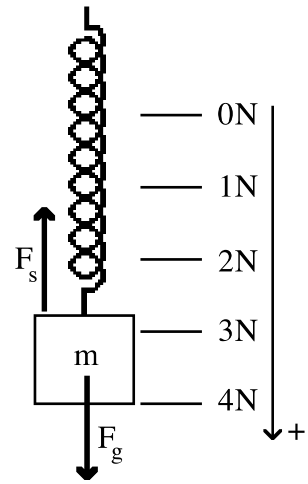
Section 36 – Systems Exhibiting SHM

The goal of this course is to answer the questions, "What do objects do?" and "Why do they do it?" Some objects exhibit simple harmonic motion or SHM. We looked carefully at the typical example of a mass vibrating at the end of a spring. We learned that any system where the acceleration is equal to minus a constant multiplied by the position is in SHM and obeys the SHM equations of motion. In this section we will look at several additional systems that exhibit SHM.

Section Outline
1. The Torsional Pendulum
2. The Physical Pendulum
3. Other Systems in SHM



For a mass at the end of a spring, Newton's Second Law gave us,

$$a(x) = -\tfrac{k}{m}x .$$

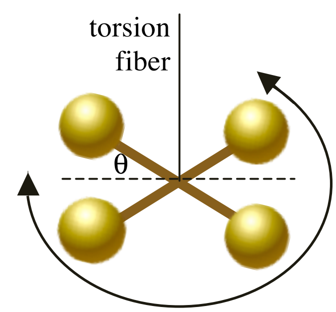
From this equation we found that the motion was oscillatory with an angular frequency equal to the root of the constants on the right hand side,

$$\omega = \sqrt{\tfrac{k}{m}} .$$

If we apply the Second Law to other systems and find that the acceleration is equal to the negative of some constants multiplied by the position, then we can follow the same logic to deduce that the motion will be SHM with an angular frequency equal to the root of the constants. Let's look at some oscillatory systems and see if they are, in fact, SHM.

1. The Torsional Pendulum

Perhaps you have seen a carriage clock such as the one shown at the left. They have a rotating pendulum. The spherical masses at the bottom are hanging from a fiber that oscillates back and forth with the spheres as shown at the right. This is due to the fact that the fiber exerts a torque on them that is proportional to the angle of rotation,



$$\tau = -\kappa\theta ,$$

where κ is called the "torsion constant." This equation is sometimes referred to as Hooke's Rule for Rotation. You can see this effect if you just hang a pencil from a string. If you rotate the pencil one way, the string will try to bring it back to where it was in equilibrium.

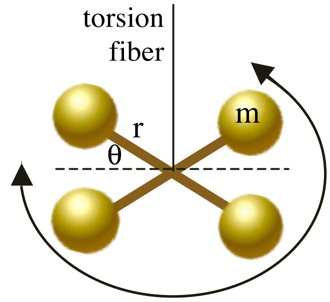
Applying the Second Law for Rotation to the hanging spheres,

$$\Sigma\tau_p = I\alpha \Rightarrow -\kappa\theta = I\alpha \Rightarrow \alpha = -\frac{\kappa}{I}\theta ,$$

where I is the rotational inertia about the center. Again we get the SHM equation where acceleration is equal to minus some constants times the displacement. The root of the constants will be the angular frequency.

Angular Frequency of a Torsional Pendulum $\omega = \sqrt{\frac{\kappa}{I}}$

Example 36.1: The clock shown above has four 50.0g masses oscillating back and forth at a radius of 2.50cm. The period of oscillation is 1.00s. Find the torsion constant of the fiber.



Given: m = 0.0500kg, r = 0.0250m, and T = 1.00s.
Find: κ = ?

The period is related to the frequency of the torsional pendulum,

$$\omega = \sqrt{\frac{\kappa}{I_p}} \Rightarrow T = 2\pi\sqrt{\frac{I_p}{\kappa}} \Rightarrow \kappa = 4\pi^2\frac{I_p}{T^2}.$$

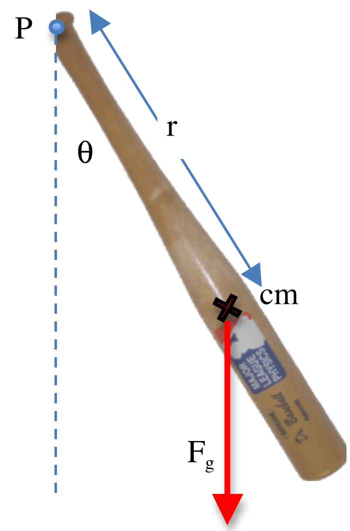
The rotational inertia of the pendulum about the center is,

$$I_p = 4mr^2.$$

The torsion constant is then,

$$\kappa = 4\pi^2\frac{4mr^2}{T^2} = 16\pi^2\frac{mr^2}{T^2} = 16\pi^2\frac{(0.0500)(0.0250)^2}{(1.00)^2} \Rightarrow \boxed{\kappa = 4.93\,\tfrac{mN\cdot m}{rad}}.$$

2. The Physical Pendulum



A simple pendulum has all its mass concentrated at a point and oscillates due to gravitational torques. Objects that don't have their mass concentrated at a point also oscillate due to gravitational torques. These systems are called "physical pendulums." For example, consider a baseball bat held near the end. Gravity provides a torque about the pivot point. Applying the Second Law for Rotation,

$$\Sigma\tau_p = I\alpha \Rightarrow -mgr\sin\theta = I_p\alpha \Rightarrow \alpha = -\frac{mgr}{I_p}\sin\theta.$$

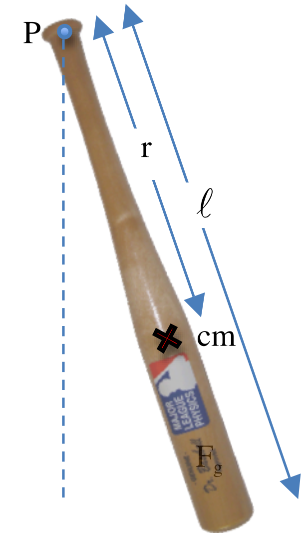
Note that r and I_p depend on where you pivot the bat. Assuming we keep the angle small, sinθ can be replaced with θ,

$$\alpha = -\frac{mgr}{I_p}\theta.$$

Once again we get the SHM equation. The acceleration is equal to minus some constants times the displacement. The root of the constants is the angular frequency.

Angular Frequency of a Physical Pendulum $\omega = \sqrt{\frac{mgr}{I_p}}$

Example 36.2: An 86.4cm long baseball bat has a mass of 0.820kg. It's center-of-mass is located 58.6cm from the handle end about which it oscillates with a period of 1.64s. (a)Find the rotational inertia of the bat about the handle. (b)Compare the result with the value for a uniform stick.



Given: ℓ = 0.864m, m = 0.826kg, r = 0.586m, and T = 1.64s.
Find:

(a)The period is related to the frequency of the physical pendulum,

$$\omega = \sqrt{\frac{mgr}{I_p}} \Rightarrow T = 2\pi\sqrt{\frac{I_p}{mgr}}.$$

Solving for I_p,

$$I_p = \frac{mgrT^2}{4\pi^2} = \frac{(0.820)(9.80)(0.586)(1.64)^2}{4\pi^2} \Rightarrow \boxed{I_p = 0.321 kg \cdot m^2}.$$

(b)For a uniform stick pivoted about its end,

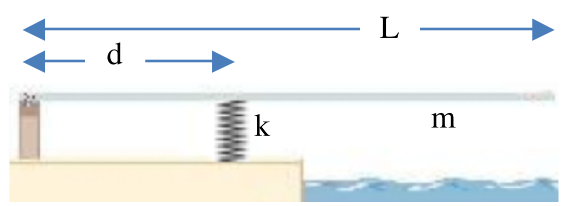
$$I = \tfrac{1}{3}m\ell^2 = \tfrac{1}{3}(0.820)(0.864)^2 \Rightarrow \boxed{I = 0.204 kg \cdot m^2}.$$

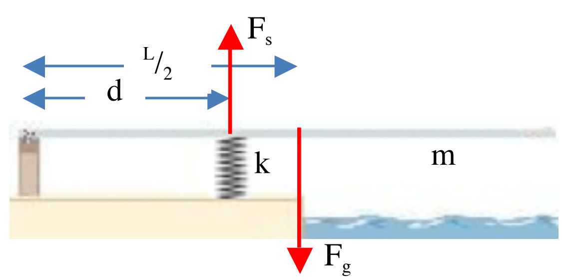
The uniform stick has a smaller rotational inertia because a baseball bat has a greater fraction of its mass at the far end, making it harder to accelerate. This is a disadvantage when you are trying to speed up the bat, but a big advantage when the collision with the ball is trying to slow it down.

3. Other Systems in SHM

We have looked at four common systems that exhibit SHM: a mass on the end of a spring, a simple pendulum, a torsional pendulum, and a physical pendulum. They all have an acceleration is equal to the negative of some constants multiplied by the position. There are many other systems that have the same property. Here is an example of one such system.

Example 36.3: A diving board of length 3.00m and mass 20.0kg has a spring (k = 3000N/m) is attached 1.00m from the left end as shown. In equilibrium it is horizontal. Find (a)the compression of the spring when the board is horizontal, (b)an expression for the torque about the left end when the board is displaced by a small angle, (c)an expression for the angular acceleration at the small angle, and (d)the frequency of oscillation when it is slightly disturbed from equilibrium.

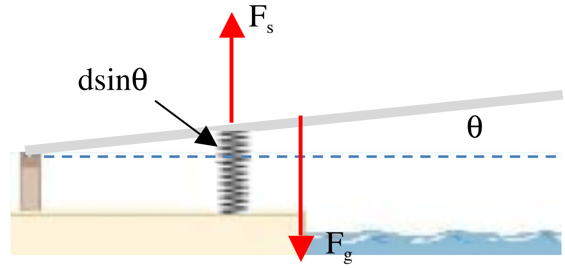




Given: L = 3.00m, d = 1.00m, m = 20.0kg, and k = 3000N/m.
Find: x_o = ?, τ = ?, α = ? and f = ?

(a)The forces that aren't at the left end are shown above. Applying the Second Law for Rotation about the left end and noting that the board is in equilibrium,

$$\Sigma\tau = I\alpha \Rightarrow F_s d - F_g \tfrac{L}{2} = 0 \Rightarrow F_s d = F_g \tfrac{L}{2}.$$



Using Hooke's Rule and the mass/weight rule,

$$kx_o = mg\tfrac{L}{2} \Rightarrow x_o = \frac{mgL}{2k}.$$

Plugging in the numbers,

$$x_o = \frac{(20)(9.8)(3)}{2(3000)} \Rightarrow \boxed{x_o = 0.0980m = 9.8cm}.$$

(b)Noting that the lever arms are slightly shorter now,

$$\Sigma\tau = F_s d\cos\theta - F_g \tfrac{L}{2}\cos\theta.$$

The spring is less compressed now so,

$$\Sigma\tau = k(x_o - d\sin\theta)d\cos\theta - mg\tfrac{L}{2}\cos\theta = (kx_o d - mg\tfrac{L}{2} - kd^2\sin\theta)\cos\theta.$$

Plugging in the expression for x_o,

$$\Sigma\tau = (k\frac{mgL}{2k}d - mg\tfrac{L}{2} - kd^2\sin\theta)\cos\theta.$$

The first two terms cancel, so the answer,

$$\Sigma\tau = -kd^2\sin\theta\cos\theta.$$

(c)Applying the Second Law and using the rotational inertia of the stick about one end,

$$\Sigma\tau = I\alpha \Rightarrow -kd^2\sin\theta\cos\theta = \tfrac{1}{3}mL^2\alpha \Rightarrow \boxed{\alpha = -\frac{3kd^2}{mL^2}\sin\theta\cos\theta}.$$

(d)The angular acceleration is not equal to the negative of a constant times the angle, so the motion is not SHM. However, for small angles cosθ→1 and sinθ→θ. The angular acceleration becomes,

$$\alpha = -\frac{3kd^2}{mL^2}\theta.$$

So, for small angles, the angular acceleration is equal to the negative of a constant times the angle, so the motion is SHM with a frequency related to the root of the constants,

$$\omega = \sqrt{\frac{3kd^2}{mL^2}} \Rightarrow f = \frac{1}{2\pi}\sqrt{\frac{3kd^2}{mL^2}}.$$

Plugging in the values,

$$f = \frac{1}{2\pi}\sqrt{\frac{3(3000)(1)^2}{20(3)^2}} \Rightarrow \boxed{f = 1.13Hz}.$$

So, we see that many systems can undergo SHM. The only requirement is the acceleration is equal to the negative of a constant times the position. The root of the constants will be the angular frequency.

Section Summary

Why do objects do what they do? Sometimes they experience simple harmonic motion. If we examine the forces or torques on the system and find the resulting acceleration is equal to the negative of a constant times the position. The root of the constants will be the angular frequency and the object will obey the equations of motion for SHM:

$$a(x) = -\omega^2 x$$
$$v(x) = \pm\omega\sqrt{A^2 - x^2}$$
$$x(t) = A\cos(\omega t + \delta)$$
$$v(t) = -\omega A\sin(\omega t + \delta)$$
$$a(t) = -\omega^2 A\cos(\omega t + \delta)$$

where A is the amplitude of the motion, ω is the angular frequency, and δ is the phase angle.

We previously examined two systems that exhibit SHM and found their angular frequencies to be,

$\omega = \sqrt{\frac{k}{m}}$ for a mass on the end of a spring and

$\omega = \sqrt{\frac{g}{\ell}}$ for a simple pendulum.

In this section we looked at two more two systems that exhibit SHM and found their angular frequencies to be,

$\omega = \sqrt{\frac{\kappa}{I_p}}$ for a torsional pendulum and

and $\omega = \sqrt{\frac{mgr}{I_p}}$ for a physical pendulum.

We also looked at less general systems and developed the ability to identify SHM and find the frequency.