Section 37 – Kepler's Rules

What is the universe made out of and how do the parts interact? That was our goal in this course. While we've learned that objects do what they do because of forces, energy, linear and angular momentum, we haven't given much thought to the nature of the interactions, such as gravity, that give rise to the forces, energies, and momenta. The purpose of the next few sections is to build a more fundamental understanding of the gravitational interaction. In the process, we will learn more about what the universe is made of matter, dark matter, and dark energy. The historical approach we'll follow will illustrate the Scientific Method where experiments lead to theories, which lead to more experiments, which continue to lead to new theories, ad infinitum.

In this section will start with a historical overview of the evolution of our understanding of gravitation. Then we'll describe the experimental work of Tycho and the theoretical efforts of Kepler.

Section Outline

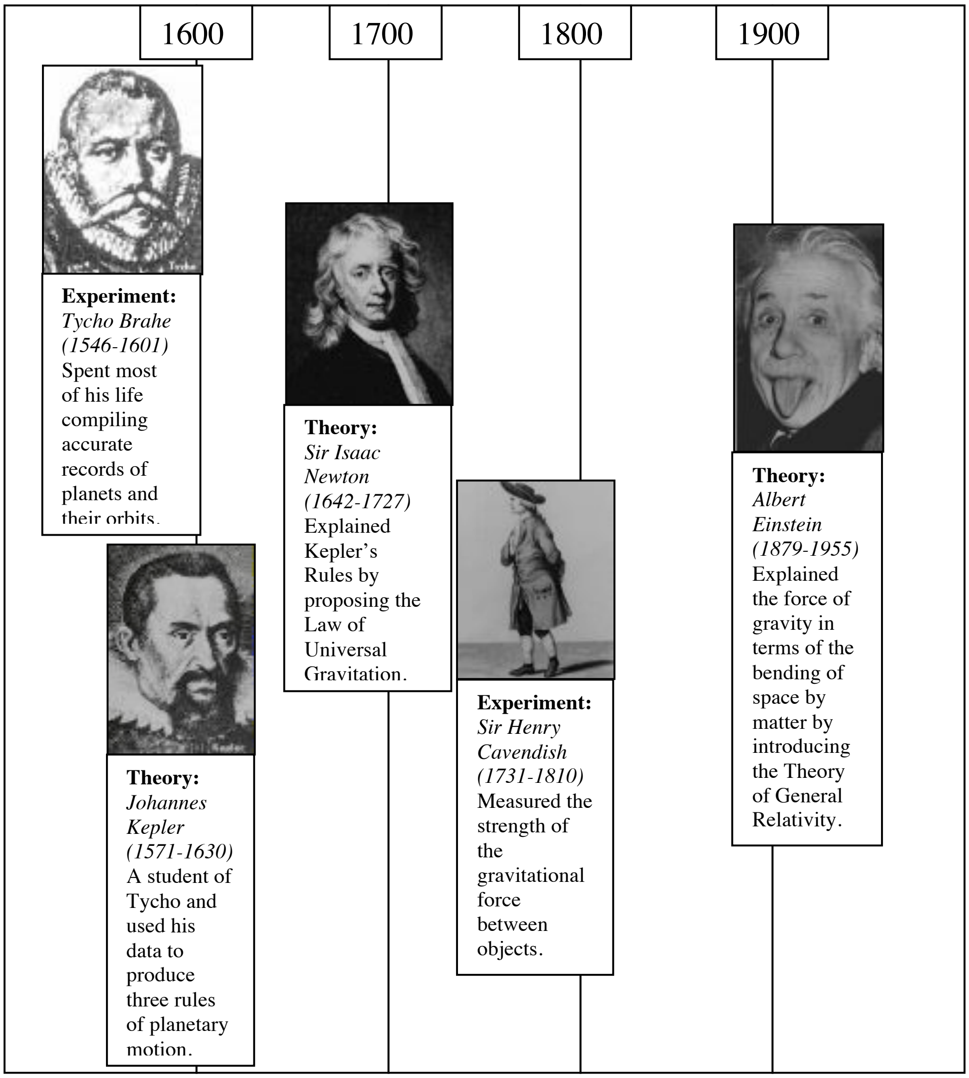
1. The History of Gravitation
2. Tycho's Data
3. Kepler's Rules of Planetary Motion
4. Newton's Law of Universal Gravitation

1. The History of Gravitation

On the next page is a timeline describing the contributions to the theory of gravitation by five major scientific figures. In the late 1500's Tycho Brahe built a remarkably accurate collection of positional data of astronomical objects. His student, Johannes Kepler used that data to create the first theory of gravitation, which he expressed as three rules of gravitational motion. About a hundred years later, Isaac Newton proposed his Laws of Motion which we have spent a good deal of time understanding. In addition, he established the Law of Universal Gravitation describing gravity in terms of a force law. Newton's Law had a constant in it the value of which he was unable to deduce. Henry Cavendish measured the value of the gravitation constant with a famous experiment. The next major contribution to our understanding of gravitation came from Einstein in the form of his Theory of General Relativity.

This history illustrates the interplay between theory and experiment that is the hallmark of the Scientific Method. The experimental data of Tycho led to the first empirical theory of gravitation by Kepler, which was improved by Newton's Law of Universal Gravitation. This law not only explained Kepler's Rules in terms of a more concise idea of force, it explained Tycho's data. However, it was missing the numerical value of a key constant, which was found by the experimental work of Cavendish. However, even with the value of the constant in hand, the force of gravity required all objects follow the same path regardless of their mass. This led Einstein to seek a deeper understanding of gravitation in his Theory of General Relativity.

The Scientific Method is a way of building knowledge based upon the verification of theories by experimental data used to develop even more explanatory theories. The history of the theory of gravity is a four hundred year long validation of the Scientific Method as a way of learning about our universe.



2. Tycho's Data

Tycho Brahe was a Danish aristocrat. He was given the best education available including attendance at the University of Copenhagen starting at the age of twelve. He learned of the prediction of a solar eclipse that occurred in his youth in August 1560 and soon devoted his life to the prediction of the heavens. He came to realize that without a quality set of data, these sorts of predictions were far to challenging and inaccurate. He soon set about building the most accurate set of data on the motion of the planets. He accomplished this goal without the aid of the telescope, which would not come into common astronomical use for another generation.

On the next page is a modern collection of solar system data for use in the examples in the next few sections.

A Collection of Solar System Data

Quantity	Sun	Mercury	Venus	Earth	Mars	Jupiter	Saturn	Uranus	Neptune	Pluto
Mean distance from sun (10^6 km)		57.9	108	150	228	778	1,430	2,870	4,500	5,900
Period of revolution (years)		0.241	0.615	1.00	1.88	11.9	29.5	84.0	165	248
Orbital speed (km/s)		47.9	35.0	29.8	24.1	13.1	9.64	6.81	5.43	4.74
Equatorial diameter (km)	1,392,000	4,880	12,100	12,800	6,790	143,000	120,000	51,800	49,500	3000?
Mass (10^{24} kg)	1,989,000	0.3333	4.87	5.974	0.639	1900	568	86.6	103	0.06?
Density (water = 1)	1.41	5.60	5.20	5.52	3.95	1.31	0.704	1.21	1.67	?
Satellites		0	0	1	2	16*	17*	15*	2*	1

Data on Selected Moons

Moon Name	Orbit Body	Orbital Radius (km)	Diameter (km)	Revolution period (Earth days)
Moon*	Earth	384,400	3476	27.322
Deimos	Mars	23,460	8	1.263
Phobos	Mars	9,270	24	0.319
Callisto	Jupiter	1,883,000	4,800	16.689
Europa	Jupiter	670,900	3126	3.551
Ganymedes	Jupiter	1,070,000	5276	7.155
Himalia	Jupiter	11,480,000	170	250.57
Io	Jupiter	421,600	1769	1.769
Thebe	Jupiter	221,900	100	0.675
Dione	Saturn	377,400	1120	2.737
Enceladus	Saturn	238,020	498	1.370
Iapetus	Saturn	3,561,300	1436	14.72
Mimas	Saturn	185,520	398	0.942
Pan	Saturn	133,630	19.32	0.5750
Tethys	Saturn	294,660	1060	1.888
Titan	Saturn	1,221,850	5150	15.945
Ariel	Uranus	191,240	1160	2.520
Miranda	Uranus	129,780	472	1.414
Oberon	Uranus	582,600	1526	13.463
Portia	Uranus	66,085	108	0.513
Puck	Uranus	86,010	154	0.762
Titania	Uranus	435,840	1580	8.706
Umbriel	Uranus	265,970	1190	4.144
Despina	Neptune	62,000	160	0.40
Galatea	Neptune	52,500	140	0.33
Larissa	Neptune	73,600	200	0.56
Nereid	Neptune	5,513,400	340	360.16
Proteus	Neptune	117,600	420	1.12
Triton	Neptune	354,800	2705	5.877
Charon	Pluto	19,640	1200	6.387

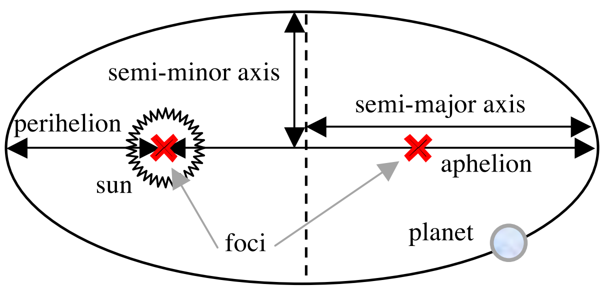
*mass 7.3477×10^{22} kg

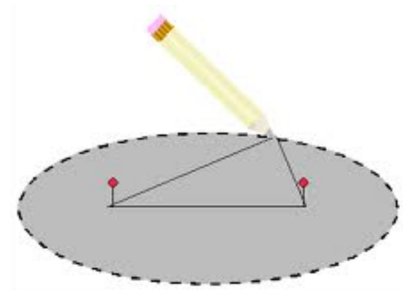
3. Kepler's Rules of Planetary Motion

Johannes Kepler was not from a wealth family but managed to acquire a university education nonetheless. In 1600, Kepler met Tycho who was sufficiently impressed with him that he allowed Kepler access to his data on the orbit of Mars. There is a long and complex story that eventually led Kepler to be appointed to replace Tycho after his unexpected death in 1601. Kepler sifted through the volumes of data collected by Tycho and from the data developed three rules to describe the motion of planets.

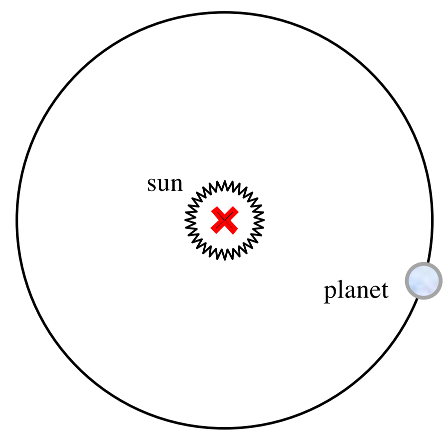
When Kepler's Laws were introduced, they were called laws because there was nothing more fundamental to explain them. Here we call them Kepler's Rules because we know they will be explained by Newton's Law of Universal Gravitation and Laws of Motion.

<u>Kepler's First Rule</u>: Planets move in elliptical orbits with the Sun at a focus.



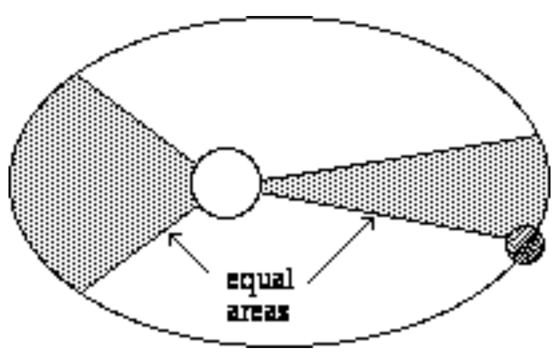


An ellipse can be drawn by tying one end of a string to each of the foci. Then, using a pencil, stretch the string taut and move it around the arc as shown at the right. The aphelion is the greatest distance between the sun and the planet while the perihelion is the shortest. The semi-major axis is the half the longest distance across the ellipse while the semi-minor axis is the half the shortest distance.



Note that for a circular orbit the foci overlap at the center and the semi-major axis, the semi-minor axis, the perihelion, and the aphelion all become equal to the radius of orbit.

<u>Kepler's Second Rule</u>: A line joining any planet to the Sun sweeps out equal areas in equal times.



The planet must therefore be moving faster when it is closer to the sun and slower when it is farther from the sun. With our understanding of mechanics we can understand this rule in two ways. When the planet is close to the sun, there is less potential energy and by the Law of Conservation of Energy, there must be more kinetic energy.

Another way to approach the issue is using the Law of Conservation of Angular Momentum. To use this law, we must assume that the gravitational force is directly along the line between the sun and the planet so there is no torque about the sun. If so, the larger the distance, the slower the planet must travel.

Here is a problem we have solved before but now you can view it in light of the equal areas law.

> *Example 37.1: Halley's comet is closest distance to the sun is 0.586AU and the farthest distance is 35.1AU. When it is farthest away it is traveling about 0.911km/s. Find the speed when it is closest to the sun.*
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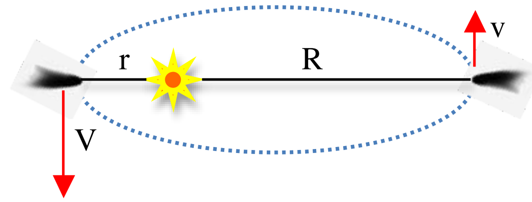
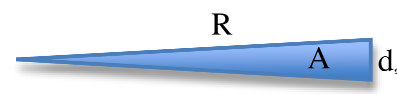
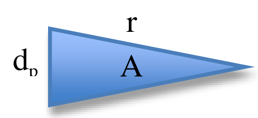
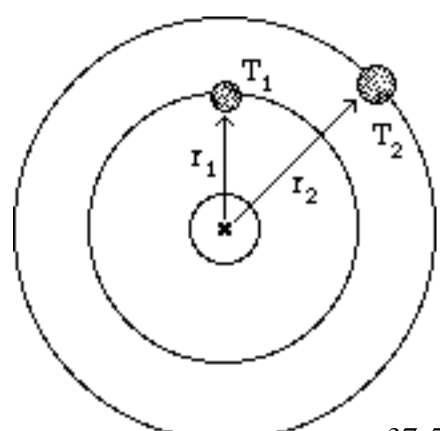
> 
>
>
> Given: r = 0.586AU, R = 35.1AU, and
> v = 0.911km/s.
> Find: V = ?
>
> Let's look at the aphelion and find the time for the comet to cover some distance d_a,
> $$d_a = vt\ .$$
> The area sweep out is shown at the right. It is the area of the triangle, which is about,
>
> 
>
>
> $$A = \tfrac{1}{2} d_a R = \tfrac{1}{2} Rvt\ .$$
> The same reasoning applies at perihelion,
>
> 
>
>
> $$A = \tfrac{1}{2} d_p r = \tfrac{1}{2} rVt\ .$$
> Using Kepler's Second Rule, the areas are equal for equal times,
> $$\tfrac{1}{2} Rvt = \tfrac{1}{2} rVt \Rightarrow Rv = rV \Rightarrow V = v\frac{R}{r}\ .$$
> This is exactly what we got using the Law of Conservation of Angular Momentum,
> $$L_o = L \Rightarrow mVr = mvR \Rightarrow V = v\frac{R}{r}\ .$$
> Plugging in the numbers,
> $$V = v\frac{R}{r} = (0.911)\frac{35.1}{0.586} \Rightarrow \boxed{V = 54.6\,\tfrac{km}{s}}.$$



Kepler's Third Rule: The square of the period of any planet about the Sun is proportional to the cube of the semi-major axis of the orbit.

If the orbit is circular, the radius is equal to the semi-major axis, so mathematically,

$$T^2 \propto r^3 \Rightarrow \frac{r^3}{T^2} = \text{constant}\ .$$

This is true for two different planets so,

$$\frac{r_1^3}{T_1^2} = \text{constant} = \frac{r_2^3}{T_2^2} \Rightarrow \left(\frac{r_1}{r_2}\right)^3 = \left(\frac{T_1}{T_2}\right)^2 .$$

Therefore, planets farther away from the sun have longer periods of orbit.

This rule puts additional constraints upon the form of the gravitational force. Applying the Second Law to one of the planets,

$$\Sigma F = ma \Rightarrow F_g = ma_c .$$

Using the expression for centripetal force and tangential speed,

$$F_g = m\frac{v_t^2}{r} = m\frac{\left(\frac{2\pi r}{T}\right)^2}{r} = 4\pi^2 m\frac{r}{T^2} .$$

Since the Third Rule requires $T^2 = Cr^3$ where C is a constant,

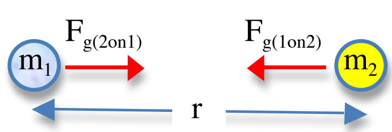
$$F_g = 4\pi^2 m\frac{r}{Cr^3} \Rightarrow F_g = \frac{4\pi^2 m}{Cr^2} .$$

So, the gravitational force must depend upon the mass of the planet and be inversely proportional to the square of the distance between the planet and the sun. So, the data of Tycho, led to the theory of Kepler summarized by his three rules. These three rules have put three constraints on the force of gravitation:

1. The force must be along the line between the sun and the planet.
2. The force depends upon the mass of the planet to the first power.
3. The force depends inversely on the square of the distance between the sun and the planet.

4. Newton's Law of Universal Gravitation

In addition to the three constraints above, Newton's Third Law puts an additional constraint of the form of the gravitational force. Consider two masses exerting gravitational forces on each other as shown. The Third Law requires the two forces to be equal in magnitude,



$$F_{g(1on2)} = F_{g(2on1)} .$$

Kepler's Rules required the force to depend upon the mass of the object feeling the force. Since both objects feel the force, the gravitational force must depend upon both masses.

With these ideas in mind, Newton proposed

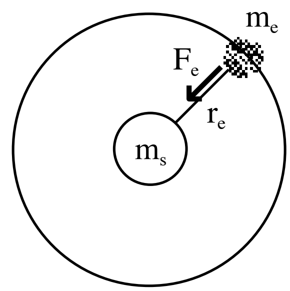
The Law of Universal Gravitation $\vec{F}_g = G\frac{m_1 m_2}{r^2}\hat{r}$

G is called the "Gravitational Constant." It's value is,

$$G = 6.67x10^{-11} \tfrac{N\cdot m^2}{kg^2} .$$

It is needed to keep the units correct and it establishes the overall size of the gravitational force. Cavendish actually measured its value as we'll discuss later. The unit vector is to remind you that the force points along the line of centers of the masses as required by Kepler's Second Rule. It falls off as one over r squared and depends upon the mass of the object feeling the force as requires by Kepler's Third Rule. Finally, it depends upon the mass exerting the force as required by Newton's Second Law.

Example 37.2: Find the gravitational force exerted by the sun on Earth and (b)the resulting acceleration of Earth.



Given: $m_s = 1.99x10^{30}$kg, $m_e = 5.97x10^{24}$kg, and $r_e = 1.50x10^{11}$m.
Find: F_e = ?, and a_e = ?

(a)Using the Law of Gravitation,

$$F_e = G\frac{m_s m_e}{r_e^2} = (6.67x10^{-11})\frac{(1.99x10^{30})(5.97x10^{24})}{(1.50x10^{11})^2} \Rightarrow$$

$$\boxed{F_e = 3.52x10^{22}N}.$$

(b)Applying Newton's Second Law to Earth,

$$\Sigma F = ma \Rightarrow F_e = m_e a_e \Rightarrow a_e = \frac{F_e}{m_e} = \frac{3.52x10^{22}}{5.97x10^{24}} \Rightarrow \boxed{a_e = 5.90x10^{-3}\,m/s^2}.$$

Even though the force on Earth is huge, the acceleration is small because of Earth's large mass.

Section Summary

We began or historical trip through the time to see the progression of the understanding of gravity. Kepler used Tycho's data to produce three rules:

Kepler's First Rule: Planets move in elliptical orbits with the Sun at a focus.

Kepler's Second Rule: A line joining any planet to the Sun sweeps out equal areas in equal times.

Kepler's Third Rule: The square of the period of any planet about the Sun is proportional to the cube of the semi-major axis of the orbit.

These three rules together with Newton's Laws of Motion required the form of the gravitational force to be described by,

$$\text{The Law of Universal Gravitation } \vec{F}_g = G\frac{m_1 m_2}{r^2}\hat{r}.$$

where the gravitation constant turns out to be,

$$G = 6.67x10^{-11}\,\tfrac{N\cdot m^2}{kg^2}.$$