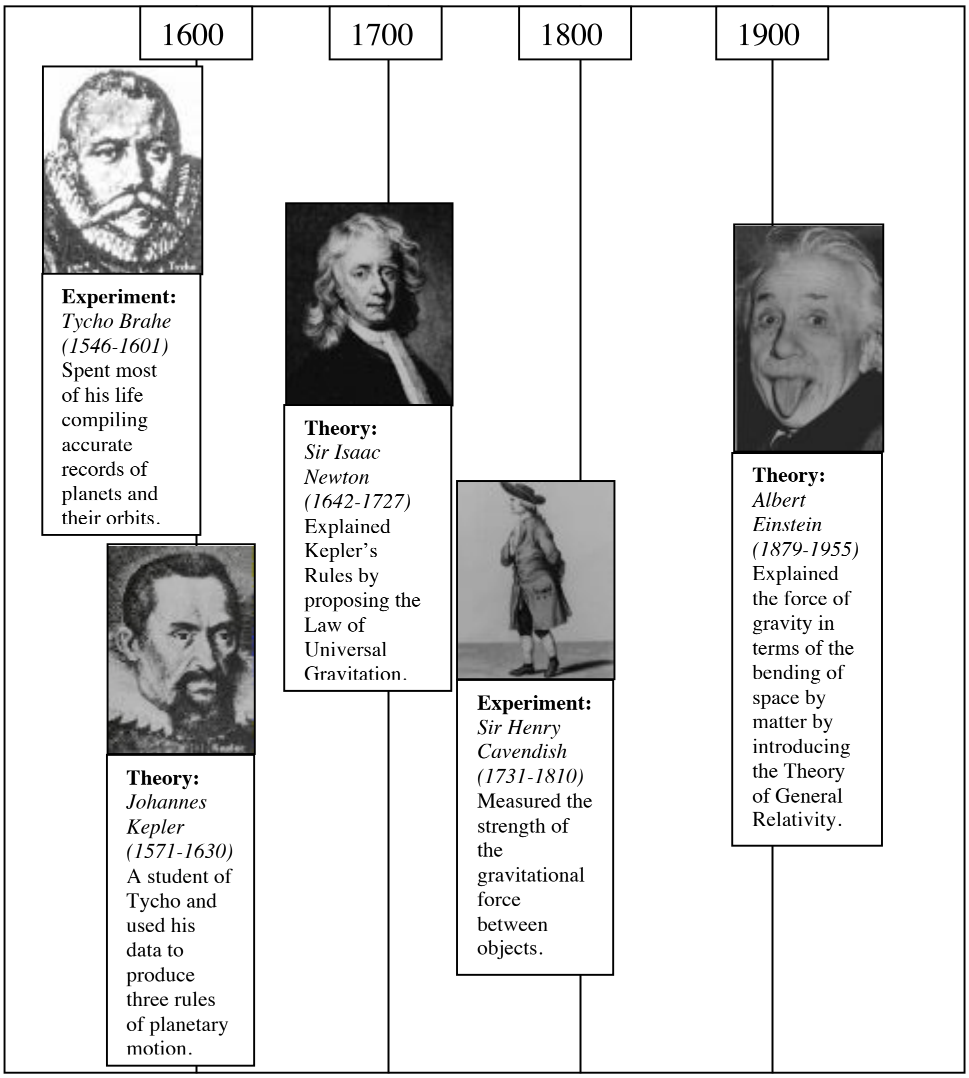
Section 38 – The Law of Gravitation

What is the universe made out of and how do the parts interact? We've learned that objects do what they do because of forces, energy, linear and angular momentum. In the last section, we began to build an understanding of the theory of gravitation by following the development of the theory through time. In this section, we'll continue our time travel as we look at the power of the Law of Universal Gravitation to explain gravitation using the idea of force.

Recall the experimental data of Tycho led to the synthesis of Kepler's Three Rules which lead Newton to proposed the Law of Universal Gravitation. So, we have made it to about 1700. In this section we'll move forward about 100 years as we learn about the next experimental contribution to our understanding of gravitation, the Cavendish Experiment.



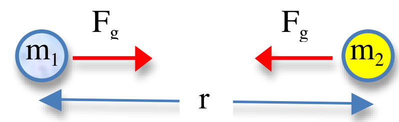
Section Outline

1. Newton's Law of Universal Gravitation
2. Explaining Kepler's Rules
3. Cavendish and the Gravitation Constant

1. Newton's Law of Universal Gravitation

Isaac Newton was perhaps the greatest scientist ever. His life was tumultuous, but he earned a degree at Trinity College in Cambridge, England. Soon afterward, the college was temporarily closed due to what became known as the Great Plague. Newton went home to Woolsthorpe and in the following two years developed the calculus, theories on optics, and the Law of Gravitation. Newton was once quoted as saying, "If I have seen further it is by standing on the shoulders of giants." As we have seen, perhaps he was referring to the contributions of Tycho and Kepler.

Consider two objects, m_1 and m_2, separated by some distance, r, as shown at the right. They exert a force on each other given by,

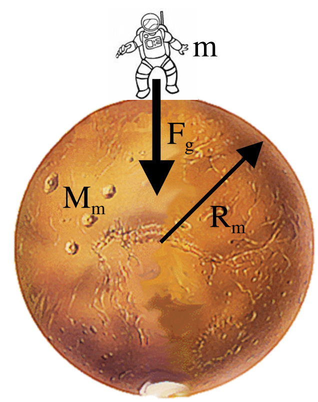


The Law of Universal Gravitation $\vec{F}_g = G\frac{m_1 m_2}{r^2}\hat{r}$

where the gravitation constant is given by $G = 6.67x10^{-11}\,\frac{N\cdot m^2}{kg^2}$.

Example 38.1: Find the acceleration due to gravity on Mars.

Given: $M_m = 6.39x10^{23}$kg and $R_m = 3.40x10^6$m
Find: g_m = ?



Applying the Second Law and the Law of Gravitation to the astronaut,

$$\Sigma F = ma \Rightarrow F_g = mg \Rightarrow G\frac{mM_m}{R_m^2} = mg \Rightarrow g = G\frac{M_m}{R_m^2}$$

Plugging in the values,

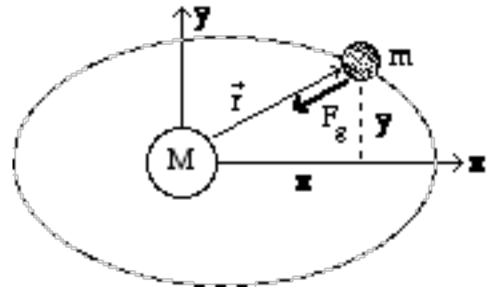
$$g = (6.67x10^{11})\frac{6.39x10^{23}}{(3.40x10^6)^2} \Rightarrow \boxed{g = 3.70m/s^2}.$$

Astronauts on Mars will feel about 40% of the gravitational acceleration as on Earth.

2. Explaining Kepler's Rules

Theories and laws are considered superior if they can explain everything that has come before in a more coherent and compact framework. The value in the Law of Gravitation is its ability to explain Kepler's Rules with the unifying ideas we have built to explain why objects do what they do. So below, we'll go through each rule and explain it using the Law of Gravitation along with our hard won knowledge of forces, energy, linear and angular momentum.

Kepler's First Rule - Elliptical Orbits:



Applying the Second Law to the planet,

$$\Sigma\vec{F} = m\vec{a} \Rightarrow \vec{F}_g = m\vec{a}.$$

Using the Law of Universal Gravitation for the force,

$$G\frac{mM}{r^2}\hat{r} = m\vec{a} \Rightarrow \vec{a} = G\frac{M}{r^2}\hat{r}.$$

The mass of the planet doesn't affect the motion of the planet.
This is the same idea we found with the Rule of Falling Bodies.
It is very strange that the force on an object depends upon its mass, while its motion does not. This oddity helped Einstein develop the Theory of General Relativity, but we'll get to that later…

Writing the unit vector in terms of the position vector,

$$\vec{a} = G\frac{M}{r^2}\cdot\frac{\vec{r}}{r} \Rightarrow \vec{a} = G\frac{M}{r^3}\vec{r}.$$

Substituting in x and y,

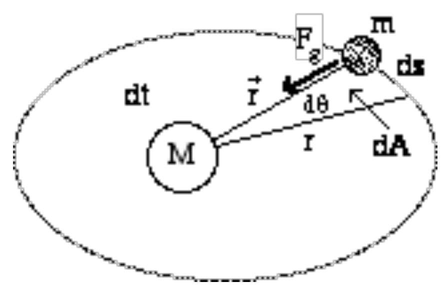
$$\left(\frac{d^2x}{dt^2}\hat{i} + \frac{d^2y}{dt^2}\hat{j}\right) = G\frac{M}{\left(x^2+y^2\right)^{3/2}}(x\hat{i} + y\hat{j}).$$

Equating the x-components and the y-components gives two horribly coupled differential equations,

$$\frac{d^2x}{dt^2} = G\frac{Mx}{\left(x^2+y^2\right)^{3/2}} \text{ and } \frac{d^2y}{dt^2} = G\frac{My}{\left(x^2+y^2\right)^{3/2}}.$$

The solution of these equations is very messy. However, the point is that with enough mathematical skill you would discover that the answer is an ellipse!

Kepler's Second Rule - Equal Areas:



For a small angle, $d\theta$, the area can be approximated by the triangle rule, one-half the base times the height,

$$dA = \tfrac{1}{2}rds.$$

Using the definition of speed,

$$v \equiv \frac{ds}{dt} \Rightarrow ds = vdt \Rightarrow dA = \tfrac{1}{2}rvdt.$$

Multiplying the top and the bottom by the mass of the planet,

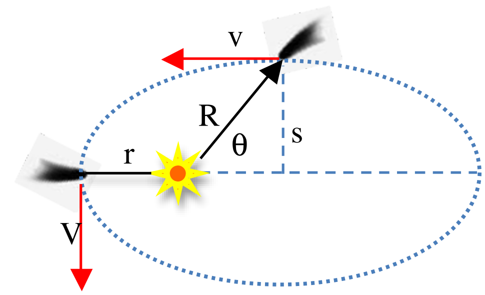
$$dA = \frac{rmv}{2m}dt.$$

The numerator is the angular momentum of the planet about the sun. The rate at which the area is swept out is then,

$$\frac{dA}{dt} = \frac{L}{2m}.$$

Since the gravitational force acts along the radius vector, the planet feels no torque due to the gravitational force. Therefore, according to the Law of Conservation of Angular Momentum, the angular momentum is constant. Since the angular momentum is constant, so is the rate at which area is swept out.

Example 38.2: The comet Giacobini–Zinner has an elliptical orbit with a perihelion of $1.56x10^{11}$m. It travels at $3.97x10^4$m/s at perihelion. It's orbit has a semi-minor axis of $6.81x10^{11}$m. Find its speed when it crosses the semi-minor axis.



Given: $r_p = 1.56x10^{11}$m, $s = 6.81x10^{11}$m and
$V = 3.97x10^4$m/s.
Find: v = ?

At perihelion the radius vector is perpendicular to the velocity vector, so the angular momentum of the comet is given by,

$$\vec{L} \equiv \vec{r} \times \vec{p} \Rightarrow L_p = rmV \ .$$

When the comet crosses the semi-minor axis, the radius vector and the velocity vector are not perpendicular,

$$\vec{L} \equiv \vec{r} \times \vec{p} \Rightarrow \vec{L}_s = \vec{R} \times m\vec{v} = m\vec{R} \times \vec{v} \ .$$

The cross product will find the part of the radius vector that is perpendicular to the velocity, which is exactly the semi-major axis,

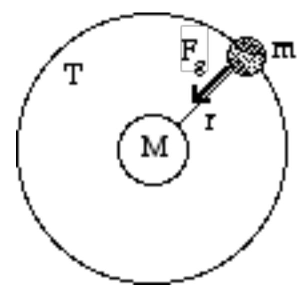
$$\vec{L}_s = m\vec{R} \times \vec{v} \Rightarrow L_s = msv \ .$$

Using the Law of Conservation of Angular Momentum,

$$L_p = L_s \Rightarrow rmV = msv \Rightarrow v = V\frac{r}{s} = (3.97x10^4)\frac{1.56}{6.81} \Rightarrow v = 9.09x10^3 \tfrac{m}{s} \ .$$

The speed of the comet increases as it nears the sun. We found this using the angular momentum, but it is also consistent with the Law of Conservation of Energy.

<u>Kepler's Third Rule - Rule of Periods:</u>



We'll prove this rule for circular orbits and save the proof for elliptical orbits for Physics 301A. Applying the Second Law to the planet,

$$\Sigma\vec{F} = m\vec{a} \Rightarrow F_g = ma \ .$$

Using the Law of Universal Gravitation and the centripetal acceleration,

$$G\frac{mM}{r^2} = m\frac{v^2}{r} \Rightarrow G\frac{M}{r} = v^2 \ .$$

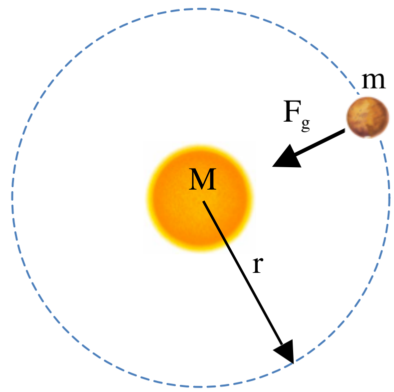
The speed is just the circumference divided by the period,

$$G\frac{M}{r} = \left(\frac{2\pi r}{T}\right)^2 \Rightarrow G\frac{M}{r} = \frac{4\pi^2 r^2}{T^2} \Rightarrow \frac{T^2}{r^3} = \frac{4\pi^2}{GM} \ .$$

So the square of the period divided by the cube of the radius is a constant for all planets in the solar system.

While Kepler's Rules are very useful for describing the motions of the planets, they don't really go very far in explaining why planets do what they do. Newton's Law of Universal Gravitation not only explains why the planets do what they do, but goes deeper to describe gravity as a universal phenomenon. Gravity is a force that is felt by all objects that have mass.

Example 38.3: Given the period of orbit of Mars is 1.88 years as well as the masses of Mars and the sun. Find its distance from the sun.



Given: T = 1.88y = 5.93x10^{7}s, m = 6.39x10^{23}kg, and M = 1.99x10^{30}kg.
Find: r = ?

Applying the Second Law to Mars,

$$\Sigma F = ma \Rightarrow F_g = ma\,.$$

Using the Law of Universal Gravitation and the centripetal acceleration,

$$G\frac{mM}{r^2} = m\frac{v^2}{r} \Rightarrow G\frac{M}{r} = v^2\,.$$

The speed is just the circumference divided by the period,

$$G\frac{M}{r} = \left(\frac{2\pi r}{T}\right)^2 \Rightarrow r = \sqrt[3]{\frac{GMT^2}{4\pi^2}}\,.$$

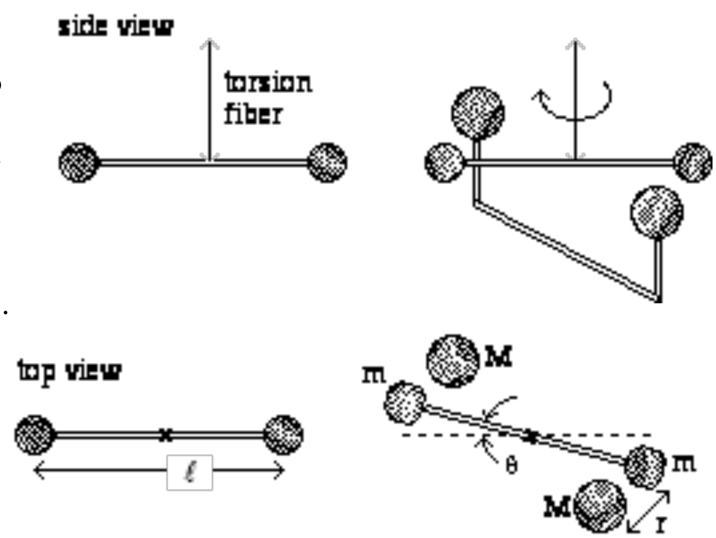
Plugging in the numerical values,

$$r = \sqrt[3]{\frac{(6.67x10^{-11})(1.99x10^{30})(5.93x10^{7})^2}{4\pi^2}} \Rightarrow \boxed{r = 2.28x10^{11}m}.$$

While Kepler's Rules were once the best we could do, we really don't need them anymore. We can solve all problems involving gravitational motion using the mechanics we have learned with the addition of the Law of Universal Gravitation.

2. Cavendish and the Gravitation Constant

Newton was not able to do the numerical examples we completed above because he had no way of knowing the value of the gravitational constant. This is were the experimental expertise of Henry Cavendish contributed to the story of gravitation.



Cavendish was a British aristocrat born in France. He attended the University of Cambridge and developed a broad interest in science. In addition to the experiment we are about to discuss, he is credited with the discovery of hydrogen.

His device for measuring G consisted of a pair of spheres hanging from a fine thread (torsion fiber). When two more sphere were brought close to the hanging pair, the pair rotated ever so slightly. By accurately measuring the rotation, Cavendish measured the gravitational force between the spheres and therefore found a value for G.

When the second pair of spheres is put in place, the original pair will find a new equilibrium by rotating toward the second set by an angle, θ. There will be two torques acting on the hanging spheres; the torque due to the gravitational forces and the torque caused by the fiber. The torque caused by the fiber is proportional to the angle,

$$\tau_f = \kappa\theta,$$

where κ is called the torsion constant. As we learned in the sections on SHM, the torsion constant can be found from the period of the oscillation about equilibrium. The torque caused by the sphere M on the sphere m due to gravity is,

$$\tau_g = \left(\tfrac{1}{2}\ell\right)F_g = G\frac{mM}{2r^2}\ell,$$

where the Law of Universal Gravitation has been used. The other two spheres M and m also exert an equal torque. Applying the Second Law for Rotation to the hanging system,

$$\Sigma\tau = I\alpha \Rightarrow \tau_f - 2\tau_g = 0 \Rightarrow \tau_f = 2\tau_g \Rightarrow \kappa\theta = 2G\frac{mM}{2r^2}\ell.$$

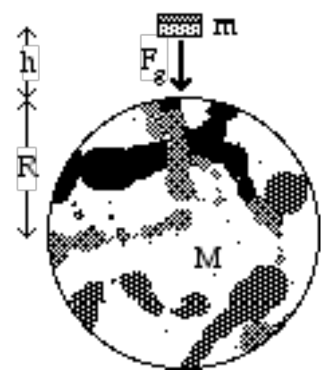
Solving for the gravitation constant,

$$G = \frac{\kappa\theta r^2}{mM\ell}.$$

All of these quantities can be measured to give a value for G. The currently accepted value is,

$$G = 6.67\text{x}10^{-11}\,\tfrac{\text{N}\cdot\text{m}^2}{\text{kg}^2}.$$

Cavendish's intent was not to find the value of G. Instead, he wanted to find the mass of Earth. What a clever person to realize that the gravitational torque between metal spheres could find the mass of Earth!



Imagine a falling object a distance h above the surface, as shown at the right. The only force acting on it is gravity. Applying the Second Law and the Law of Universal Gravitation,

$$\Sigma F = ma \Rightarrow F_g = mg \Rightarrow G\frac{mM}{(R+h)^2} = mg \Rightarrow M = \frac{g(R+h)^2}{G}.$$

Since the height is much smaller than the radius of Earth,

$$M \approx \frac{gR^2}{G}.$$

The acceleration due to gravity was known since Galileo and the radius of Earth was known by measuring the curvature. Once Cavendish measured G the mass of Earth became known. Putting in the values,

$$M = \frac{(9.8)(6.4\text{x}10^6)^2}{6.7\text{x}10^{-11}} = 6.0\text{x}10^{24}\,\text{kg}.$$

Section Summary

We continued our historical trip through the time to advance our understanding of the gravitational interaction. We showed that

$$\text{The Law of Universal Gravitation } \vec{F}_g = G\frac{m_1 m_2}{r^2}\hat{r}$$

along with the rest of our knowledge of mechanics was sufficient to explain all three of Kepler's Rules. In addition, we showed how Cavendish "weighed Earth" by designing an experiment to measure

$$\text{The Gravitational Constant } G = 6.67x10^{-11}\,\tfrac{N\cdot m^2}{kg^2}\,.$$