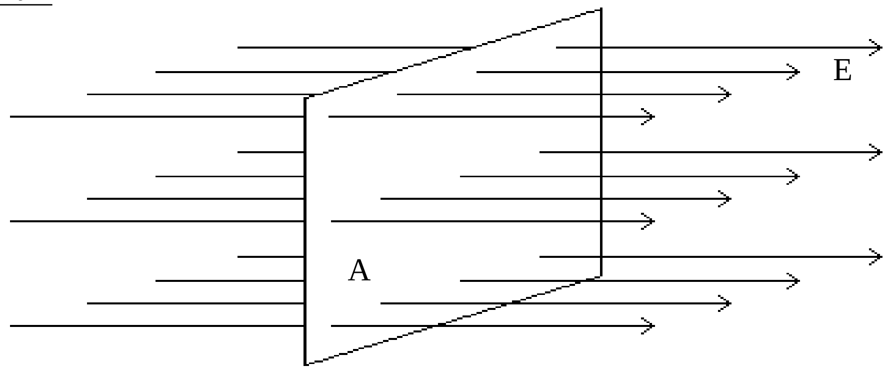
Chapter 24 - Gauss' Law

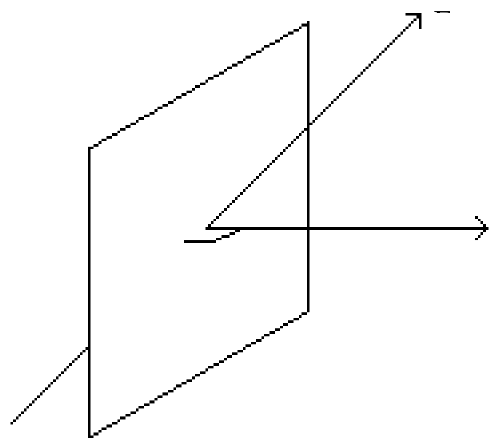
Problem Set #3 - due:
Ch 24 - 2, 3, 6, 10, 12, 19, 25, 27, 35, 43, 53, 54

Lecture Outline
1. The Definition of Electric Flux
2. Gauss' Law
3. The Behavior of Conductors
4. Examples Using Gauss' Law

1. The Definition of Electric Flux

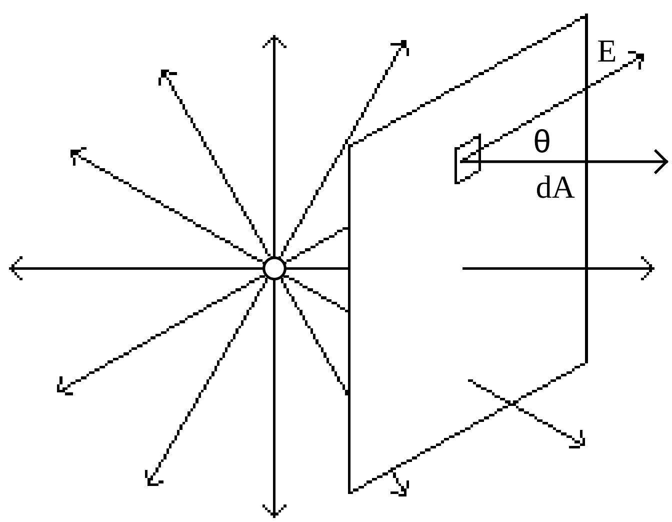
Recall that the strength of the field is proportional to the density of field lines. The field can be thought of as the number of lines per unit area. The number of lines through an area is called the "flux." If a constant field is perpendicular to the surface, then the flux is given by $\Phi = EA$.





If a constant field is at an angle θ from the normal to the surface, then the flux is given by $\Phi = EA\cos\theta = \vec{E} \cdot \vec{A}$.

For non-constant fields, such as the field due to a point charge, the flux through the surface is the sum of the normal component of the field over each small section of the surface where the flux can be considered constant.



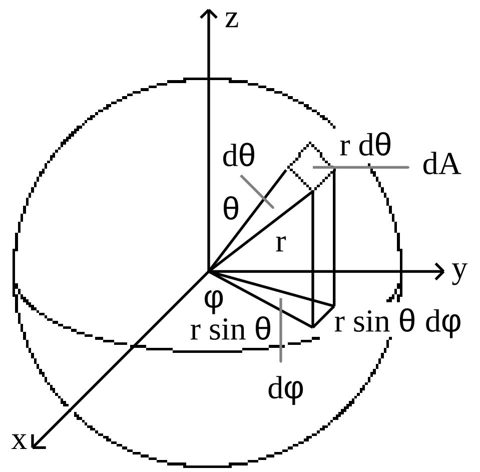
$$\Phi \equiv \int \vec{E} \cdot d\vec{A}$$ The Definition of Electric Flux

<u>Example 1:</u> Find the flux that exits a sphere centered at the origin due to a point charge also at the origin.

A small surface area on a sphere is, $d\vec{A} = r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$. Using the field due to the point charge, $\vec{E} = k\frac{q}{r^2}\hat{r}$, the flux can be calculated from its definition, $\Phi \equiv \int \vec{E} \cdot d\vec{A}$,

$$\Phi \equiv \int k\frac{q}{r^2}\hat{r} \cdot r^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$$

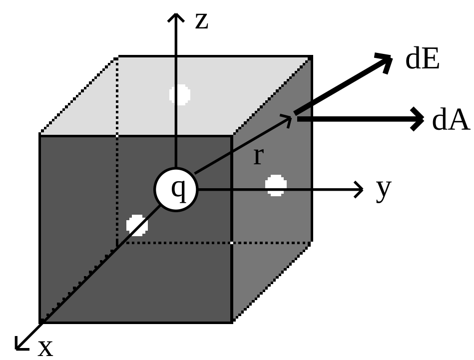
$$\Phi = k\,q\int_0^{\pi} \sin\theta \, d\theta \int_0^{2\pi} d\varphi = 4\pi k q = \frac{q}{\varepsilon_o}$$



2. Gauss' Law

<u>Example 2:</u> Find the flux due to a point charge over the surface of a cube.

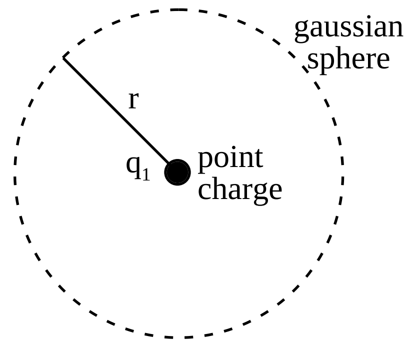
Again start with the definition of flux, $\Phi \equiv \int \vec{E} \cdot d\vec{A}$, and the field of a point charge, $\vec{E} = k\frac{q}{r^2}\hat{r}$. This time however, the electric field and the area element are not always parallel and the integral is nearly impossible to complete. But, we know that the number of field lines that leave the charge is not effected by the shape of some imaginary surface that surrounds the charge. That is, the same number of field lines exit the cube as exit the sphere in example 1 so the flux must again equal $\Phi = \frac{q}{\varepsilon_o}$.



The flux through any closed surface is proportional to the charge contained inside.

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o} \quad \text{Gauss's Law}$$

Beyond the Mechanical Universe (vol. 29 frame 28270 [Ch 17])



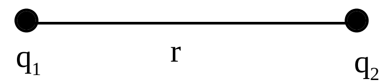
Example 3: Starting with Gauss's Law derive Coulomb's Rule.

The field due to a point charge q_1 can be found by applying Gauss's Law to an imaginary gaussian sphere of radius r centered on the charge,

$$\oint \vec{E} \cdot d\vec{A} = \frac{q}{\varepsilon_o} = \frac{q_1}{\varepsilon_o}.$$

By the spherical symmetry of the problem, the field must be radial and constant on the gaussian sphere so,

$$\oint \vec{E} \cdot d\vec{A} = EA = E4\pi r^2 = \frac{q_1}{\varepsilon_o} \qquad E = \frac{q_1}{4\pi\varepsilon_o r^2} = k\frac{q_1}{r^2}.$$



Now suppose a charge q_2 is in the field of q_1 a distance r away as shown at the right. The force on the q_2 can be found, by the definition of electric field.

$$E \equiv \frac{F}{q} \qquad k\frac{q_1}{r^2} = \frac{F}{q_2} \qquad F = k\frac{q_1 q_2}{r^2}$$

which is precisely Coulomb's Rule. Since it can be derived from Gauss' Law, I don't call it "Coulomb's Law," instead I call it a rule.

In summary, Gauss's Law is usually used in either of two ways:

1) Given the field and the surface then enclosed charge can be found.
2) Given the enclosed charge and sufficient symmetry to choose a convenient surface, then the field can be found.

3. The Behavior of Conductors

Conductor: A material in which electrons are completely free to move in response to applied electric fields.

In a conductor the electrons will move until they find a place where the field is zero. Therefor, the field inside a conductor will always be zero, if you wait long enough for the electrons to find these places. Typically, this takes microseconds.

Beyond the Mechanical Universe (vol. 29 Ch 20)

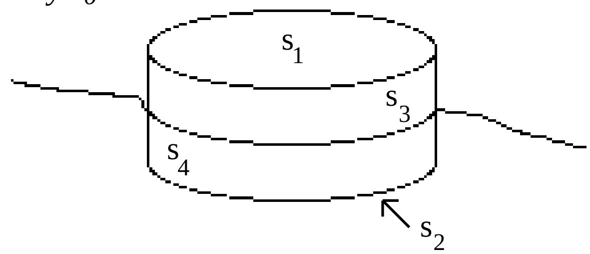
Example 4: Given a conductor of arbitrary shape with a total charge, Q, show that this excess charge must be located on the surface.

Imagine a surface just inside the real surface of the conductor over which we will apply Gauss' Law. Since this "Gaussian surface" is inside a conductor the field at it is zero everywhere

$$\oint E \cdot dA = 0 \qquad \frac{q_{encl}}{\varepsilon_o} = 0 \qquad q_{encl} = 0.$$

The gaussian surface can be made arbitrarily close to the real surface and the enclosed charge must still be zero. Therefor, the excess charge, Q, must be on the surface and not inside.

<u>Example 5:</u> Show that the electric field at the surface of any conductor is perpendicular to the local surface and is equal to the local charge density, σ, divided by ε_o.



The field along the surface of the conductor must be zero. Otherwise, the free charges inside would feel a force and move until the field along the surface was zero. Therefor, the field is perpendicular to the surface.

Starting with Gauss' Law, $\oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$,

we choose a gaussian surface that straddles the surface of the conductor. We choose it in such a way that surface 1 is parallel to the local surface of the conductor. The total flux leaving the surface is just the sum of the fluxes leaving the four surfaces shown.

$$\oint \vec{E} \cdot d\vec{A} = \int_{S_1} \vec{E} \cdot d\vec{A} + \int_{S_2} \vec{E} \cdot d\vec{A} + \int_{S_3} \vec{E} \cdot d\vec{A} + \int_{S_4} \vec{E} \cdot d\vec{A}$$

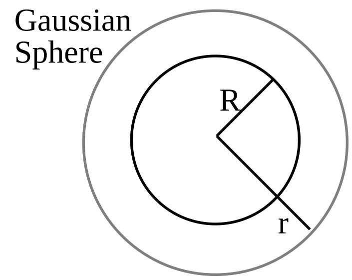
Over surfaces 2 and 4 the electric field is zero because they are inside a conductor. To find the field at the surface we need surface 1 to approach the surface of the conductor which makes surface 3 infinitely small so no flux can leave it. The total flux leaving the gaussian surface is just the flux through surface 1.

$$\int_{S_1} \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o} = \frac{1}{\varepsilon_o}\int \sigma \, dA \qquad E = \frac{\sigma}{\varepsilon_o}$$

To summarize the behavior of conductors,

1) E=0 everywhere inside.
2) Excess charge stays on the surface.
3) $E = \frac{\sigma}{\varepsilon_o}$ and perpendicular to the surface just outside.

<u>Example 6:</u> Find the electric field inside and outside a spherical conductor of radius, R, with a charge, Q. Sketch a graph of E vs. r.



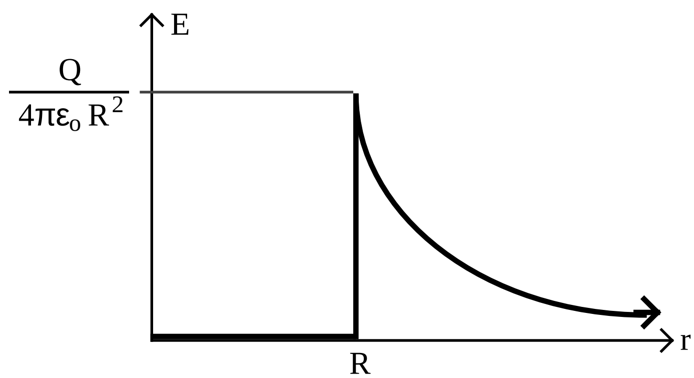
Inside the conductor the field is always zero. To find the field outside, notice that by symmetry the field must always point directly away from the center of the sphere and can only depend on the distance from the sphere. If we choose a gaussian surface that is a concentric sphere of radius, r, the field will be constant over its surface and always point directly outward parallel with the local dA vector. Find the flux leaving this surface:

$$\oint \vec{E} \cdot d\vec{A} = \oint E\, dA$$ because E and dA are parallel.

$$\oint E\, dA = E \oint dA = E\, 4\pi r^2$$ because the field is constant over this sphere.

According to Gauss' Law this must equal the enclosed charge, Q, over ε_o.

$$E\, 4\pi r^2 = \frac{Q}{\varepsilon_o} \qquad E = \frac{Q}{4\pi\varepsilon_o r^2} = k\frac{Q}{r^2}$$ The same as if all the charge were at the center.



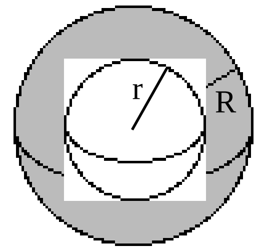
4. Examples Using Gauss' Law

Gauss' Law may be used in two ways:

1)Given the field and the surface then enclosed charge can be found.

2)Given the enclosed charge and sufficient symmetry then the field can be found.

<u>Example 7:</u> A total charge, Q, is uniformly distributed throughout a non-conducting sphere of radius, R. Find the electric field inside and out. Sketch E vs. r.

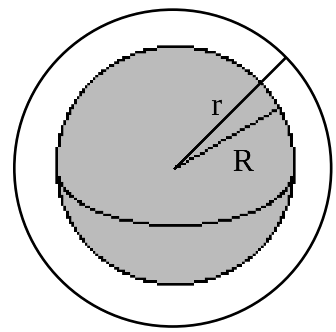


The spherical symmetry means that E will be constant over any concentric gaussian sphere and E will point radially (parallel with dA). Therefor the flux integral in Gauss' Law is, $\oint \vec{E} \cdot d\vec{A} = E\,A = E(4\pi r^2)$.

For a gaussian sphere with r<R the charge enclosed is proportional to the fraction of the volume of the real sphere that the gaussian sphere occupies.

$q_{encl} = \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} Q = \frac{r^3}{R^3} Q$. Applying Gauss' Law, $\oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$,

$$E(4\pi r^2) = \frac{r^3}{R^3}\frac{Q}{\varepsilon_o} \qquad E = k\frac{Qr}{R^3} \quad r < R$$



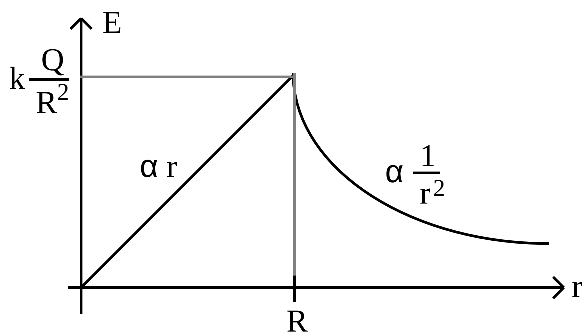
For a gaussian sphere with r>R the charge enclosed is just the total charge, Q.

Applying Gauss' Law,

$$\oint \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o} \qquad E(4\pi r^2) = \frac{Q}{\varepsilon_o} \qquad E = k\frac{Q}{r^2} \quad r > R$$

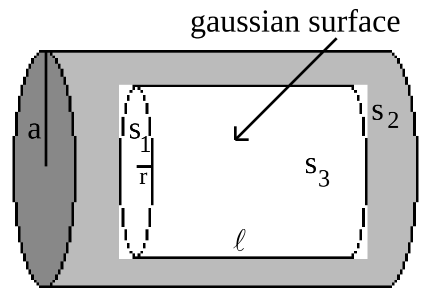
This field is just the field due to a point charge Q at the origin.

Notice the equations for the fields inside and outside agree at r=R as they must.



<u>Example 8:</u> *A very long non-conducting cylinder of radius, a, has a linear charge density, λ, uniformly distributed throughout its volume. Find the electric field as a function of the distance from the axis, r, and sketch a graph of E vs. r.*

By symmetry the electric field must point radially outward and it can only depend on r. Therefor, the best gaussian surface is a concentric cylinder of radius, r.



The flux integral in Gauss' Law can be broken up into three parts. One for each surface shown,

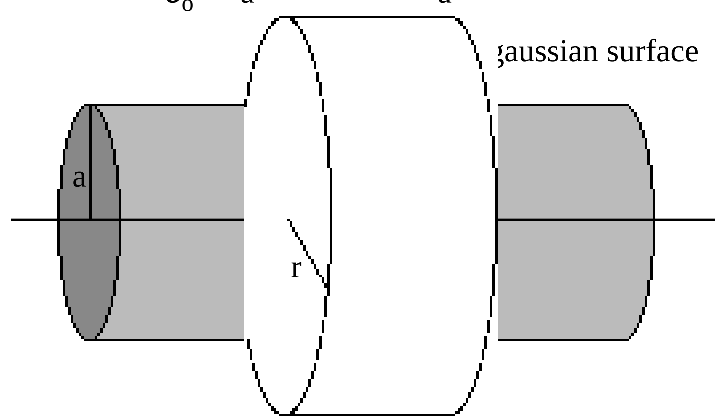
$$\oint \vec{E}\cdot d\vec{A} = \int_{s_1} \vec{E}\cdot d\vec{A} + \int_{s_2} \vec{E}\cdot d\vec{A} + \int_{s_3} \vec{E}\cdot d\vec{A}.$$

The integrals over s_1 and s_2 are both zero because the field is radial which means that no flux exits these faces (E is perpendicular to dA). Over s_3 E is constant and parallel to dA so that integral is, $\int_{s_3} \vec{E}\cdot d\vec{A} = EA = E2\pi r\ell$.

For r<a the charge enclosed in the gaussian surface is, $q_{encl} = (\lambda\ell)\dfrac{\pi r^2\ell}{\pi a^2\ell} = \lambda\ell\dfrac{r^2}{a^2}$.

Applying Gauss' Law, $\oint \vec{E}\cdot d\vec{A} = \dfrac{q_{enclosed}}{\varepsilon_o}$ $\quad E(2\pi r\ell) = \dfrac{1}{\varepsilon_o}\lambda\ell\dfrac{r^2}{a^2}$ $\quad E = 2k\lambda\dfrac{r}{a^2}\quad r < a$

For r>a the charge enclosed in the gaussian surface is, $q_{encl} = \lambda\ell$.

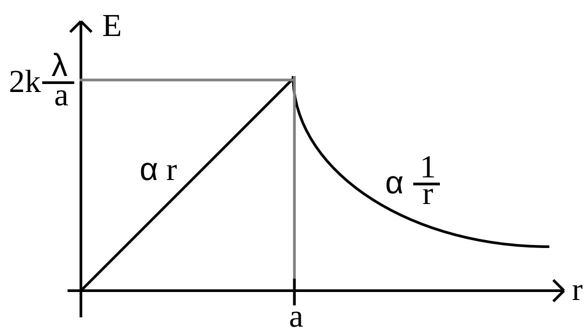


Applying Gauss' Law,

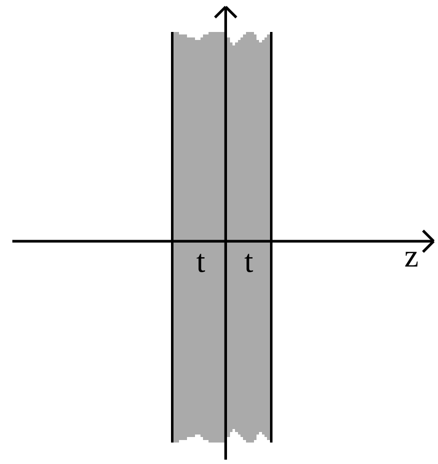
$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$$

$E(2\pi r\ell) = \dfrac{1}{\varepsilon_o}\lambda\ell \quad E = 2k\dfrac{\lambda}{r}\quad r > a$

Note that the field falls off as $\dfrac{1}{r}$ not $\dfrac{1}{r^2}$ as you might expect. The equations for the fields inside and outside agree at r=a as they must.



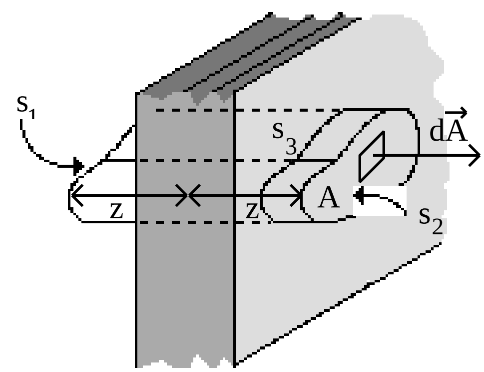
<u>*Example 9:*</u> *A large metal plate of thickness, 2t, has a uniform charge density, σ. Find the electric field as a function of the distance from the center of the plate, z. Find the charge distribution and sketch E vs. z.*



By symmetry the E-field can only be a function of z and it must point directly away from the plate. Inside the plate the field is zero because it is a conductor. Outside we can choose the gaussian surface shown.

The surfaces s_1 and s_2 are the same arbitrary shape, parallel with the surface of the plate, and equal distances from it. The gaussian surface is completed with the surface s_3 which is everywhere perpendicular to the surface of the plate. The flux leaving the gaussian surface is,

$$\oint \vec{E}\cdot d\vec{A} = \int_{s_1} \vec{E}\cdot d\vec{A} + \int_{s_2} \vec{E}\cdot d\vec{A} + \int_{s_3} \vec{E}\cdot d\vec{A}\,.$$



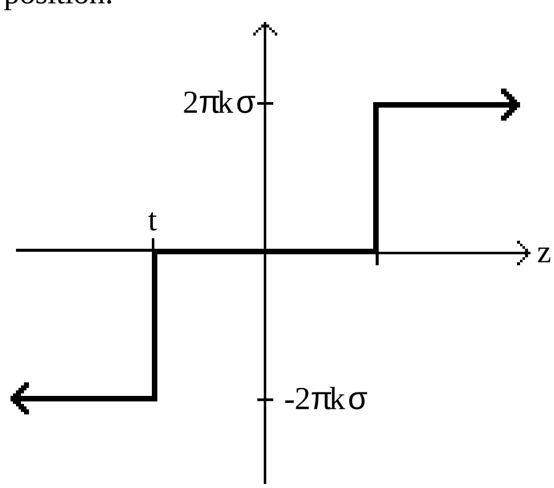
Since the field is only along the z-axis the integral over s_3 is zero. The magnitude of the field can only depend on z, so the fields at s_1 and s_2 are equal and constant so the integrals are straightforward,

$$\oint \vec{E}\cdot d\vec{A} = EA + EA + 0 = 2EA$$

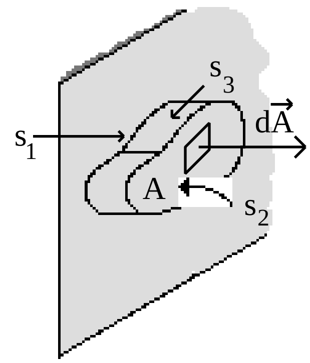
The charge enclosed in the gaussian surface is, $q_{encl} = \sigma A$. Applying Gauss' Law,

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o} \qquad 2EA = \frac{1}{\varepsilon_o}\sigma A \qquad E = \frac{\sigma}{2\varepsilon_o} = 2\pi k\sigma \quad |z| > t$$

This agrees with the result from integrating up the charge distribution in the previous chapter. Sketching the field versus position:



To find the charge distribution on the plate, consider the gaussian surface shown at the right. It is the same as before except the surface s_1 is now just inside the conductor so the field over it is zero. The flux is,



$$\oint \vec{E}\cdot d\vec{A} = \int_{s_1} \vec{E}\cdot d\vec{A} + \int_{s_2} \vec{E}\cdot d\vec{A} + \int_{s_3} \vec{E}\cdot d\vec{A} = 0 + EA + 0 = EA = 2\pi k\sigma A$$

Applying Gauss' Law,

$$\oint \vec{E}\cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o} \qquad 2\pi k\sigma A = 4\pi k q_{enclosed} \qquad \frac{q_{enclosed}}{A} = \frac{\sigma}{2}$$

Half the charge is on each surface of the plate (not surprising!).

Chapter 24 - Summary

The Definition of Electric Flux $\Phi \equiv \int \vec{E}\cdot d\vec{A}$

Gauss' Law $\oint \vec{E}\cdot d\vec{A} = \frac{q_{enclosed}}{\varepsilon_o}$

The Behavior of Conductors,

1) E=0 everywhere inside.
2) Excess charge stays on the surface.
3) $E = \frac{\sigma}{\varepsilon_o}$ and perpendicular to the surface just outside.