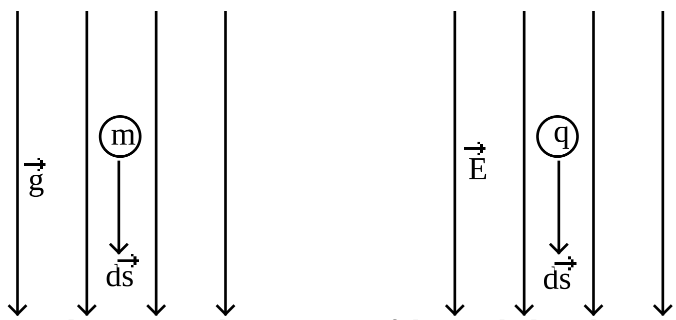
Chapter 25 - Electric Potential

Problem Set #4 - due:
Ch 25 - 6, 7, 12, 17, 22, 28, 34, 36, 41, 50, 56, 63

The Law of Conservation of Energy is the single most useful law of physics. In this chapter we will develop the insights needed to make use of energy conservation to understand the ideas associated electrical phenomena.

Lecture Outline
1. The Potential Energy of Charges in Electric Fields
2. The Definition of Electric Potential
3. The Potential Due to Point Charges
4. The Potential Due to Continuous Charge Distributions
5. Finding the Electric Field from the Potential

1. The Potential Energy of Charges in Electric Fields

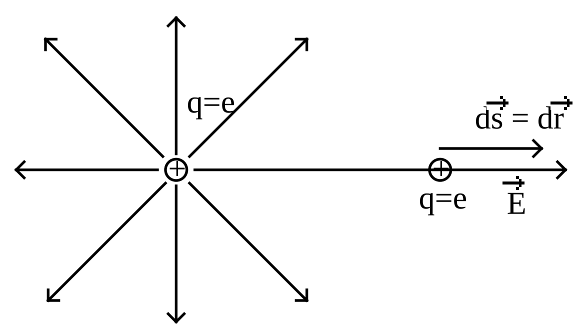


Recall the definition of potential energy as the negative of the work done by a conservative force, $\Delta U \equiv -W_c$. Using the definition of work, $\Delta U = -\int \vec{F}\cdot d\vec{s}$.

Recall how this went with gravity, $\Delta U_g = -\int \vec{F}_g \cdot d\vec{s} = -\int m\vec{g}\cdot d\vec{s} = -m\int \vec{g}\cdot d\vec{s}$ in general.
For a constant gravitational field, g, ΔU_g=-mgh.
It goes the same way with electricity, $\Delta U_e = -\int \vec{F}_e \cdot d\vec{s} = -\int q\vec{E}\cdot d\vec{s} = -q\int \vec{E}\cdot d\vec{s}$.

$$\Delta U = -q\int \vec{E}\cdot d\vec{s} \qquad \text{Electric Potential Energy}$$

Example 1: Find the change in electric potential energy when two protons initially 0.100nm apart are completely separated.



The electric potential energy is, $\Delta U = -q\int \vec{E}\cdot d\vec{s}$.

The field is due a point charge and the motion is along the same direction, $\hat{r}$.

$$\Delta U = -e\int_R^\infty k\frac{e}{r^2}dr = -ke^2\left[-\frac{1}{r}\right]_R^\infty = -k\frac{e^2}{R}$$

$$\Delta U = -9x10^9\,\frac{\left(1.6x10^{-19}\right)^2}{0.100x10^{-9}} = \underline{\underline{-2.3x10^{-18}\,J}}$$

Note that $\Delta U = U(\infty) - U(R)$. It is customary to set $U(\infty) \equiv 0$ so we can say that the original potential energy of the protons was $U(R) = \underline{\underline{+2.3x10^{-18} J}}$. These protons originally had a lot of potential energy when close together and are quite happy to fly apart when given the chance.

2. The Definition of Electric Potential

The electric potential is defined as the potential energy per unit of charge.

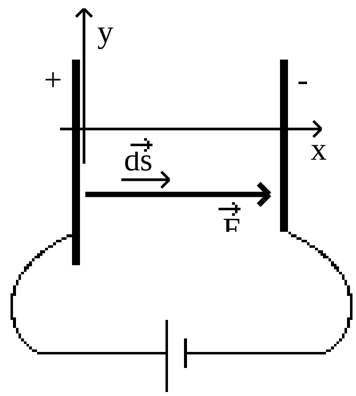
$$\Delta U \equiv q\,\Delta V \quad \text{The Definition of Electric Potential}$$

The units of potential must be $[\Delta V] = \frac{[\Delta U]}{[q]} = \frac{\text{joules}}{\text{coulomb}}$. By definition, $\frac{1\text{ joule}}{\text{coulomb}} = 1\text{ Volt} = 1\text{V}$

The electric potential energy $\Delta U = -q\int \vec{E}\cdot d\vec{s}$ shows that the relationship between the field and the potential is,

$$\Delta V = -\int \vec{E}\cdot d\vec{s} \quad \text{Calculation of the Potential from the E-field}$$

Example 2: Find the voltage required on a set of parallel plates 10.0cm apart to create a field of 1000N/C.

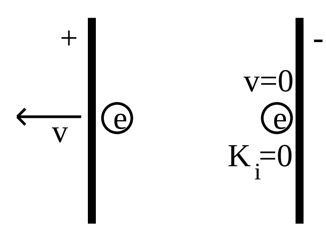


The relationship between the field and the potential is, $\Delta V = -\int \vec{E}\cdot d\vec{s}$.

For parallel plates the field is constant so,

$$\Delta V = -\int E\hat{i}\cdot dx\hat{i} = -E\int_0^d dx = -Ed = -(1000)(.100) = \underline{\underline{-100V}}$$

Example 3: Find the kinetic energy that an electron released from the negative plate would have just as it reaches the positive plate.



The change in the potential energy of the electron can be found from the definition of potential,

$$\Delta U \equiv q\,\Delta V = (-1.6x10^{-19}C)(+100V) = -1.6x10^{-17}J$$

Applying the Law of Conservation of Energy,

$$\Delta U + \Delta K = 0 \qquad \Delta U = -\Delta K = -K_f + K_i = -K_f$$

$$K_f = -\Delta U = +1.6x10^{-17} J$$

There is a more convenient unit for energy in problems like this.

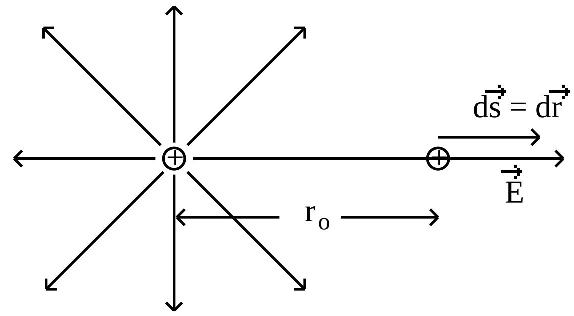
Define 1 electron·volt = 1 eV ≡ $1.6x10^{-19}$ J

$$\Delta U = (-1e)(+100V) = -100eV \text{ and } K_f = -\Delta U = +100eV$$

3. The Potential Due to Point Charges

Starting with the relationship between the field and the potential, $\Delta V = -\int \vec{E} \cdot d\vec{s}$ and using the field of a point charge, $\Delta V = -\int k\frac{q}{r^2}\hat{r} \cdot d\vec{s} = -kq\int_{r_o}^{\infty}\frac{dr}{r^2} = -k\frac{q}{r_o}$.

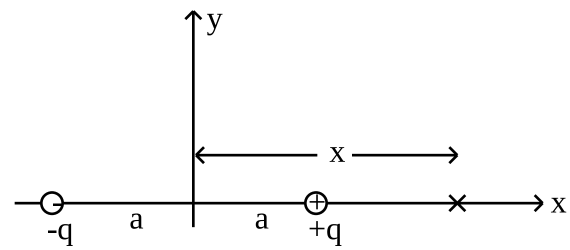
Note that $\Delta V = V(\infty) - V(r_o)$. If we define $V(\infty)=0$, we get the potential due to a point charge, $V(r_o) = k\frac{q}{r_o}$



$$V = k\frac{q}{r} \quad \text{The Potential Due to a Point Charge}$$

Example 4: Find the potential due to a distant dipole along its axis.

Since potential is a scalar quantity we can just add up the potentials due to the individual point charges.



$$V = V_+ + V_- = k\frac{q}{x-a} + k\frac{-q}{x+a} = kq\left\{\frac{1}{x-a} - \frac{1}{x+a}\right\}$$

$V = kq\frac{2a}{x^2-a^2} = k\frac{p}{x^2-a^2}$ where p=2qa the dipole moment.

For x >> a, $V = k\frac{p}{x^2}$

Example 5: Find the energy required to assemble all the charges in example 4 assuming that there is a charge Q at the position x.

From the definition of potential, $\Delta U \equiv q\,\Delta V$, we can find the potential energy needed to bring in each charge to create this distribution. Choose the +q to be the first one. It can be moved into position using no energy because there are no other charges around yet ($\Delta V=0$). Choose the -q next,

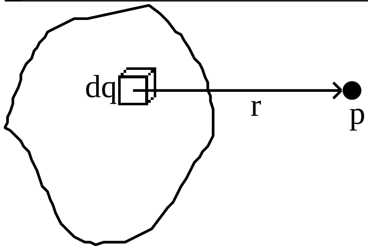
$\Delta U_{dipole} = -q\,V_+ = -q\,k\frac{q}{2a} = -k\frac{q^2}{2a}$ where V_+ is the potential difference felt by the -q.

Next bring in the charge Q.

$$\Delta U_Q = Q\,(V_+ + V_-) = QV_{dipole} = Q\,k\frac{p}{x^2} = k\frac{2aqQ}{x^2}$$

The total energy required is, $\Delta U = \Delta U_{dipole} + \Delta U_Q = k\frac{2aqQ}{x^2} - k\frac{q^2}{2a} = kq\left\{\frac{2aQ}{x^2} - \frac{q}{2a}\right\}$.

4. The Potential Due to Continuous Charge Distributions

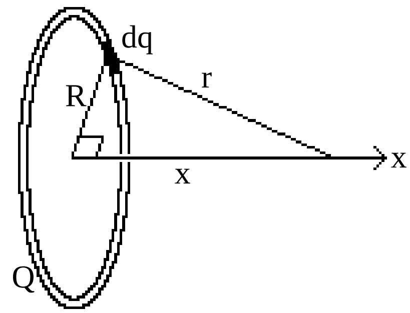


The potential due to the point charge, dq, is $dV = k\frac{dq}{r}$.
To find the potential due to the entire charge distribution we need to sum over the dq's. That is, we need to integrate:

$$\int dV = k\int\frac{dq}{r} \qquad V = k\int\frac{dq}{r}.$$

$V = k\int\frac{dq}{r}$	The Electric Potential Due to a Continuous Charge Distribution

Example 6: Find the potential due to a ring of charge, Q, and radius, R, a distance, x, from the center along the axis.

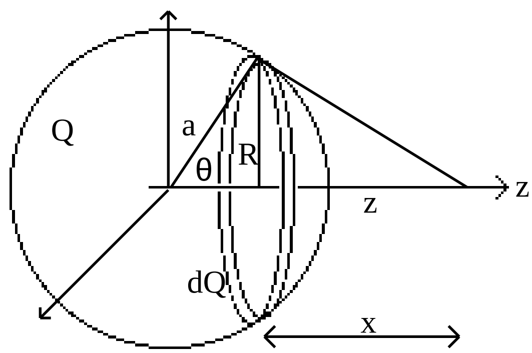


Choose a small portion of the ring as a point charge, dq, and use the potential due to a point charge, $dV = k\frac{dq}{r}$. Since r is constant for all dq's on the ring, the integral is straightforward,

$$\int dV = k\int\frac{dq}{r} \qquad V = k\int\frac{dq}{r} = \frac{k}{r}\int dq = \frac{k}{r}Q.$$

In terms of R and x, $V = k\frac{Q}{\left(R^2 + x^2\right)^{1/2}}$.

Example 7: Find the potential due to a spherical shell of radius, a, and charge, Q, a distance, z, from the center of the shell.



Use the rings from example 6 as the dq's and sum their potentials to find the potential due to the entire sphere. By trigonometry, $x = z - a\cos\theta$, so the potential, dV, due to the ring at an angle θ is,

$$dV = k\frac{dQ}{\left(R^2 + [z - a\cos\theta]^2\right)^{1/2}}.$$

Note that $R = a\sin\theta$, so,

$$dV = k\frac{dQ}{\left(a^2\sin^2\theta + z^2 - 2za\cos\theta + a^2\cos^2\theta\right)^{1/2}} \quad dV = k\frac{dQ}{\left(a^2 + z^2 - 2za\cos\theta\right)^{1/2}}$$

Since the charge is distributed uniformly over the surface,

$$\frac{dQ}{Q} = \frac{\text{area of the strip}}{\text{area of the sphere}} = \frac{2\pi R\, ad\theta}{4\pi a^2} = \frac{2\pi a^2\sin\theta\, d\theta}{4\pi a^2} \qquad dQ = \frac{Q}{2}\sin\theta\, d\theta.$$

Now dV can be written entirely in terms of θ, $dV = k\frac{Q\sin\theta\, d\theta}{2\left(a^2 + z^2 - 2za\cos\theta\right)^{1/2}}$.

The integration can be completed by letting $u = a^2 + z^2 - 2za\cos\theta \quad du = 2za\sin\theta\, d\theta$. So,

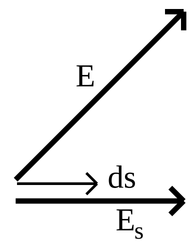
$$\int_0^V dV = \frac{kQ}{4az}\int_{\theta=0}^{\pi} u^{-1/2}\, du \qquad V = \frac{kQ}{4az}\left[2\left(a^2 + z^2 - 2za\cos\theta\right)^{1/2}\right]_0^{\pi}$$

Finally, $V = \frac{kQ}{2az}\left[\sqrt{(a+z)^2} - \sqrt{(a-z)^2}\right]$.

There are two cases: $V = \begin{cases} \text{if } z > a \quad V = \frac{kQ}{2az}\left[(a+z) - (z-a)\right] = k\frac{Q}{z} \\ \text{if } z < a \quad V = \frac{kQ}{2az}\left[(a+z) - (a-z)\right] = k\frac{Q}{a} \end{cases}$

Notice that the potential is constant inside the sphere which is consistent with the constant field we expect. It falls off as the potential of a point charge outside.

5. Finding the Electric Field from the Potential



To calculate the potential from the field we use $\Delta V = -\int \vec{E}\cdot d\vec{s}$.

To go the other way around we write, $dV = -\vec{E}\cdot d\vec{s}$.

The right hand side is the component of E along ds times ds itself, $dV = -E_s\, ds$.

Dividing both sides by ds gives the component of E along ds, $E_s = -\frac{dV}{ds}$.

$$E_s = -\frac{dV}{ds} \quad \text{Calculation of the E-field from the Potential}$$

Since $\vec{E} = E_x\hat{i} + E_y\hat{j} + E_z\hat{k}$ we can write $\vec{E} = -\left(\frac{\partial V}{\partial x}\hat{i} + \frac{\partial V}{\partial y}\hat{j} + \frac{\partial V}{\partial z}\hat{k}\right) \equiv -\nabla V$

Example 8: Find the axial field due to a dipole.

From example 4, $V = k\frac{p}{x^2}$. The electric field along the x-axis can be found by applying

$$E_s = -\frac{dV}{ds} \qquad E_x = -\frac{dV}{dx} = 2k\frac{p}{x^3} \text{ as we got before!}$$

Example 9: Find the axial field due to a ring of charge Q.

From example 6, $V = k\frac{Q}{\left(R^2 + x^2\right)^{1/2}}$. The electric field along the x-axis can be found by applying

$$E_s = -\frac{dV}{ds} \qquad E_x = -\frac{dV}{dx} = -kQ\left\{-\frac{1}{2}\left(R^2 + x^2\right)^{-3/2}(2x)\right\} = k\frac{Qx}{\left(R^2 + x^2\right)^{3/2}}$$

as we got before!

Chapter 25 - Summary

Electric Potential Energy $\Delta U = -q\int \vec{E} \cdot d\vec{s}$

The Definition of Electric Potential $\Delta U \equiv q\,\Delta V$

Calculation of the Potential from the E-field $\Delta V = -\int \vec{E} \cdot d\vec{s}$

The Potential Due to a Point Charge $V = k\frac{q}{r}$

The Electric Potential Due to a Continuous Charge Distribution $V = k\int \frac{dq}{r}$

Calculation of the E-field from the Potential $E_s = -\frac{dV}{ds}$