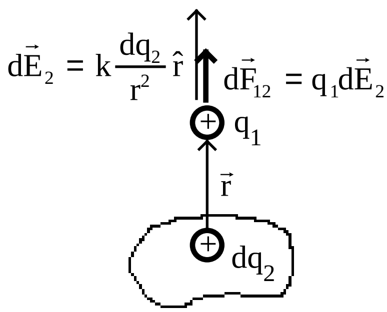
Chapter 30 - Magnetic Fields Due to Currents

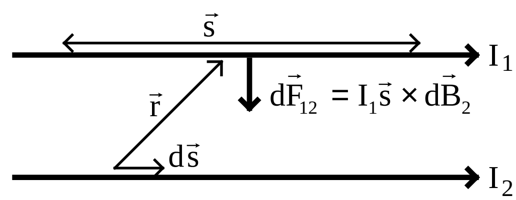
Problem Set #9 - due:
Ch 30 - 3, 12, 17, 19, 26, 37, 38, 47, 51, 54, 57, 64, 72

Lecture Outline
1. The Biot-Savart Rule
2. Ampere's Law

Now that we know the effects of magnetic fields on currents we can discuss the causes of magnetic fields. If currents feel the magnetic force, Newton's Third Law requires they must exert it as well. The purpose of this chapter is to learn to calculate magnetic fields caused by currents. The Biot-Savart Rule is a recipe that will always work. Ampere’s Law, while more fundamental, can be used to find fields only in cases where the current distribution is highly symmetric.

1. The Biot-Savart Rule





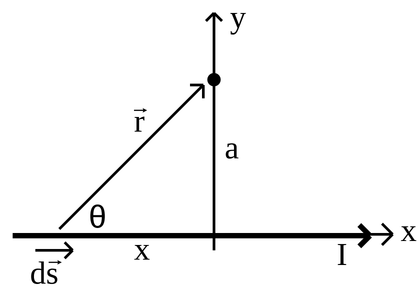
The electric force on a charge q_1 is found from the field due to all the other charges. We begin by finding the field due to the point charge dq_2, which is $d\vec{E}_2 = k\frac{dq_2}{r^2}\hat{r}$. Next, the force on q_1 due to dq_2 is found using definition of electric field, $d\vec{F}_{12} = q_1 d\vec{E}_2$. Then integrate all the contributions from the rest of the charge distribtion. The same proceedure would work for the magnetic force on I_1 due to a distribution of current I_2. The magnetic force on the current I_1 in the magnetic field $d\vec{B}_2$ caused by a small current element in I_2 is, $d\vec{F}_{12} \equiv I_1\vec{s}\times d\vec{B}_2$ which means that the magnetic equivalent of the electric charge element dq is, $q_1 \rightarrow I_1\vec{s}\times \quad dq \rightarrow Id\vec{s}\times \quad dq_2 \rightarrow I_2 d\vec{s}\times$. Using an analogy to the field due to the point charge dq_2, $d\vec{E}_2 = k\frac{dq_2}{r^2}\hat{r} \rightarrow d\vec{B}_2 = k_m\frac{I_2 d\vec{s}\times}{r^2}\hat{r} \qquad \vec{B}_2 = k_m\int\frac{I_2 d\vec{s}\times\hat{r}}{r^2}$. This is called the Biot-Savart Rule.

$$\vec{B} = \frac{\mu_o}{4\pi}\int\frac{Id\vec{s}\times\hat{r}}{r^2} \quad \text{The Biot-Savart Rule}$$

The magnetic constant is usually replaced with $k_m = \frac{\mu_o}{4\pi} \equiv 10^{-7}\frac{T\ m}{A}$.

<u>*Example 1:*</u> *Find the magnetic field due to a long straight wire carrying a current I.*

Starting with the Biot-Savart Rule, $\vec{B} = \frac{\mu_o}{4\pi}\int \frac{Id\vec{s} \times \hat{r}}{r^2}$.



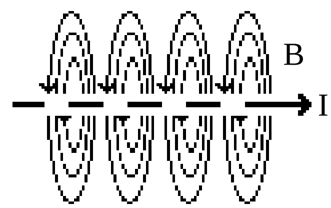
Note $|d\vec{s} \times \hat{r}| = \sin\theta dx$ and the field points out of the page.

The magnitude of the field is, $B = \frac{\mu_o}{4\pi}\int I\frac{\sin\theta}{r^2}dx$.

I is constant, $r = a\csc\theta$ and $x = -a\cot\theta$ $\quad dx = a\csc^2\theta d\theta$.

$$B = \frac{\mu_o I}{4\pi}\int_{\theta=0}^{\pi}\frac{\sin\theta}{a^2\csc^2\theta}a\csc^2\theta d\theta = \frac{\mu_o I}{4\pi a}\int_{\theta=0}^{\pi}\sin\theta d\theta = \frac{\mu_o I}{4\pi a}\ 2 = \frac{\mu_o I}{2\pi a}$$

At this point the field is out of the page. In general, the field lines form rings around the wire.



It is convenient to know the magnetic field due to a long straight wire.

$B = \frac{\mu_o I}{2\pi r}$	The Magnetic Field Due to a Long Straight Wire

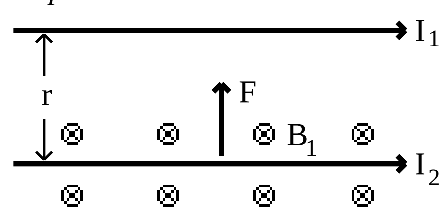
Beyond the Mechanical Universe (vol. 35 Ch 8,9,12)

<u>*Example 2:*</u> *Use the result of example 1 to find the force between two parallel currents.*

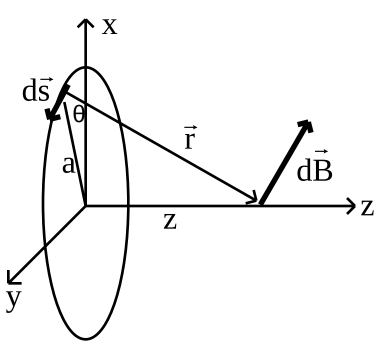
Starting with the definition of magnetic field, $\vec{F} \equiv I_2\vec{\ell} \times \vec{B}_1$.

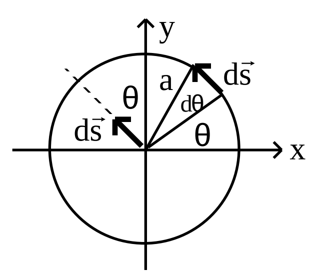
Since the direction is right, just find the magnitude, $F = I_2\ell B_1$. Using the result of example 1,

$F = I_2\ell\frac{\mu_o I_1}{2\pi r}$ $\qquad F = \frac{\mu_o}{2\pi}\frac{I_1 I_2}{r}\ \ell$ just like before.



<u>*Example 3:*</u> *Find the magnetic field on the axis of a circular loop of radius a and magnetic dipole moment* $\vec{\mu}$.





Use the Biot-Savart Rule $\vec{B} = \frac{\mu_o}{4\pi}\int\frac{Id\vec{s} \times \hat{r}}{r^2} = \frac{\mu_o I}{4\pi}\int\frac{d\vec{s} \times \vec{r}}{r^3}$.

Notice that $\vec{r} = -a\cos\theta\hat{i} - a\sin\theta\hat{j} + z\hat{k}$ and $d\vec{s} = ad\theta\left(-\sin\theta\hat{i} + \cos\theta\hat{j}\right)$.

$$d\vec{s}\times\vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -a\sin\theta d\theta & a\cos\theta d\theta & 0 \\ -a\cos\theta & -a\sin\theta & z \end{vmatrix}$$

$$d\vec{s}\times\vec{r} = \left[az\cos\theta d\theta\hat{i} + az\sin\theta d\theta\hat{j} + a^2\left(\sin^2\theta + \cos^2\theta\right)d\theta\hat{k}\right]$$

Since $r^3 = \left(a^2+z^2\right)^{3/2}$ and is constant,

$$\vec{B} = \frac{\mu_o I}{4\pi}\frac{1}{\left(a^2+z^2\right)^{3/2}}\left[az\hat{i}\int_0^{2\pi}\cos\theta d\theta + az\hat{j}\int_0^{2\pi}\sin\theta d\theta + a^2\hat{k}\int_0^{2\pi}d\theta\right]$$

Doing the integrals,

$$\vec{B} = \frac{\mu_o I}{4\pi}\frac{1}{\left(a^2+z^2\right)^{3/2}}\left[0+0+a^2\hat{k}\,2\pi\right] \qquad \vec{B} = \frac{\mu_o I}{2}\frac{a^2}{\left(a^2+z^2\right)^{3/2}}\hat{k}$$

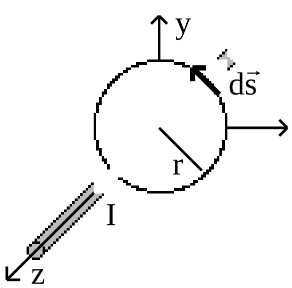
In terms of the dipole moment, $\vec{B} = \dfrac{\mu_o}{2\pi}\dfrac{\vec{\mu}}{\left(a^2+z^2\right)^{3/2}}$.

2. Ampere's Law

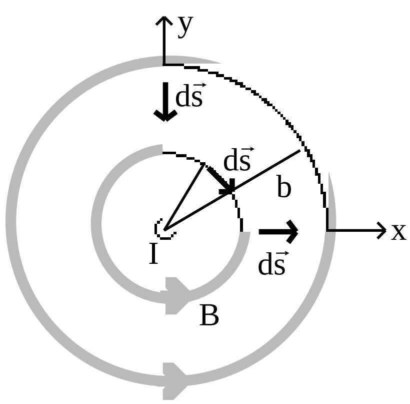
Beyond the Mechanical Universe (vol. 35 Ch 19,20)

Ampere's Law states that the sum of the magnetic field along any closed path is proportional to the current that passes through. Mathematically,

$$\oint\vec{B}\cdot d\vec{s} = \mu_o i_{enclosed} \quad \text{Ampere's Law}$$



In order to understand this idea, consider a circular path around a wire that passes through the center and is perpendicular to the plane of the circular path. We know that the magnetic field due to the wire is, $B = \frac{\mu_o I}{2\pi r}$ and it is parallel to $ds = rd\theta$. So, $\oint\vec{B}\cdot d\vec{s} = \oint\frac{\mu_o I}{2\pi r}\,rd\theta = \frac{\mu_o I}{2\pi}\oint d\theta = \frac{\mu_o I}{2\pi}\,2\pi = \mu_o I$ consistent with Ampere's Law.



If we change the path so that it doesn't contain the current as shown on the right. The path needs to be broken up into four parts,

$$\oint\vec{B}\cdot d\vec{s} = \int_x\vec{B}\cdot d\vec{s} + \int_b\vec{B}\cdot d\vec{s} + \int_y\vec{B}\cdot d\vec{s} + \int_a\vec{B}\cdot d\vec{s}.$$

Along the x and y axes $\vec{B}$ $d\vec{s}$ $\quad \int_x\vec{B}\cdot d\vec{s} = \int_y\vec{B}\cdot d\vec{s} = 0$.

Along b, $B = \frac{\mu_o I}{2\pi b}$ and it is parallel to $ds = bd\theta$ so,

$$\int_b\vec{B}\cdot d\vec{s} = \int_b\frac{\mu_o I}{2\pi b}\,bd\theta = \frac{\mu_o I}{2\pi}\int_b d\theta = \frac{\mu_o I}{2\pi}\,\frac{\pi}{2} = \frac{\mu_o I}{4}.$$

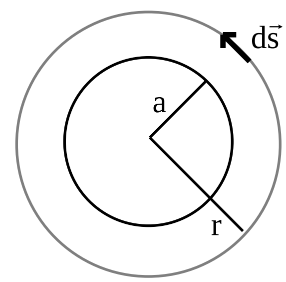
Similarly along a, $\int_a \vec{B}\cdot d\vec{s} = -\int_a \frac{\mu_o I}{2\pi a}\, a d\theta = -\frac{\mu_o I}{2\pi}\int_a d\theta = -\frac{\mu_o I}{2\pi}\left(\frac{\pi}{2}\right) = -\frac{\mu_o I}{4}$. Putting it all together,

$\oint \vec{B}\cdot d\vec{s} = 0 + \frac{\mu_o I}{4} + 0 - \frac{\mu_o I}{4} = 0$ also consistent with Ampere's Law.

Like Gauss's Law, Ampere's Law can be used to find the magnetic field of a sufficiently symmetric current distribution.

<u>Example 4:</u> Find the magnetic field inside and outside of a wire of radius a carrying a uniform current I. Sketch the field as a function of r the distance from the center.

Outside the wire choose the circular path shown. Along this path the symmetry requires B must be constant and it can only point along ds so,

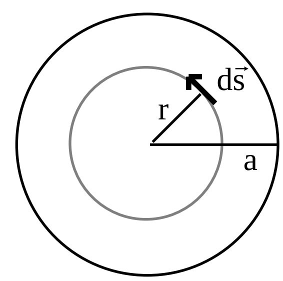


$\oint \vec{B}\cdot d\vec{s} = \oint B ds = B\oint ds = B(2\pi r)$.

The enclosed current is the entire current I. Applying Ampere's Law,

$\oint \vec{B}\cdot d\vec{s} = \mu_o i_{enclosed} \Rightarrow B(2\pi r) = \mu_o I \Rightarrow B = \frac{\mu_o I}{2\pi r}$. This is the same result as for an infinitely thin wire.

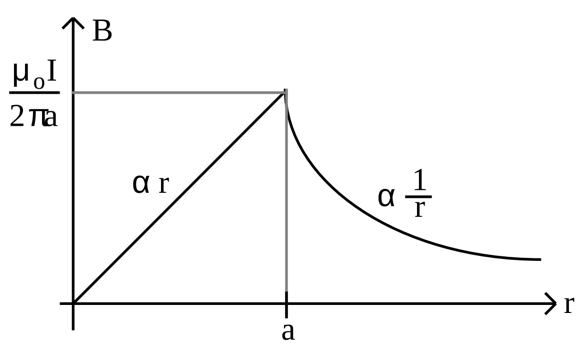
Inside the wire choose the circular path shown. Along this path the symmetry requires B must be constant and it can only point along ds so again,



$\oint \vec{B}\cdot d\vec{s} = \oint B ds = B\oint ds = B(2\pi r)$.

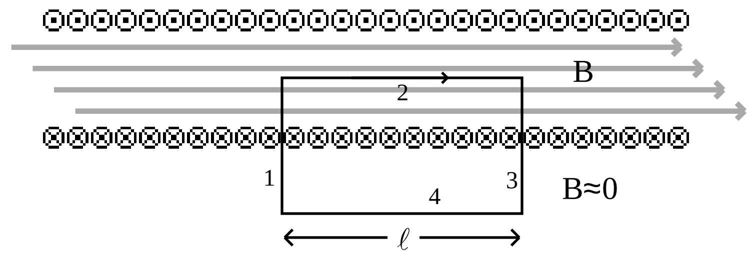
The enclosed current is $i_{enclosed} = \frac{\pi r^2}{\pi a^2} I$. Applying Ampere's Law,

$\oint \vec{B}\cdot d\vec{s} = \mu_o i_{enclosed} \Rightarrow B(2\pi r) = \mu_o \frac{\pi r^2}{\pi a^2} I \Rightarrow B = \frac{\mu_o I}{2\pi a}\left(\frac{r}{a}\right)$. The field grows linearly until the surface of the wire is reached. The graph of B vs. r is shown below.



Example 5: Wire 1.00mm in diameter is wound around a 5.00cm diameter 1.00m long tube to create a solenoid. The wire carries a current of 100mA. Estimate the magnetic field in the center.

Apply Ampere's Law for the path indicated. The current enclosed is $i_{enclosed} = Ni$ where N is the number of turns inside the path and i is the current in the wire. The path integral of the field can be broken into four parts, one for each side of the path.



$\oint \vec{B}\cdot d\vec{s} = \int_1 \vec{B}\cdot d\vec{s} + \int_2 \vec{B}\cdot d\vec{s} + \int_3 \vec{B}\cdot d\vec{s} + \int_4 \vec{B}\cdot d\vec{s}$. Along side 4 the field is close to zero. Along sides 1 and 3 the field is perpendicular to the path so, $\oint \vec{B}\cdot d\vec{s} = 0 + \int_2 \vec{B}\cdot d\vec{s} + 0 + 0 = B\ell$.

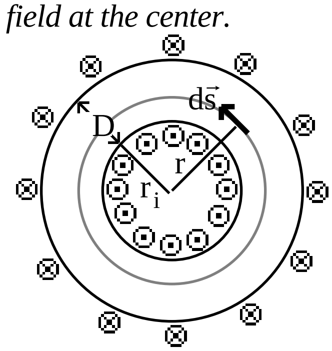
Assuming that the solenoid is so long that symmetry requires the magnetic field to be constant along the path. Using Ampere's Law,

$$\oint \vec{B}\cdot d\vec{s} = \mu_o i_{enclosed} \qquad B\ell = \mu_o Ni \qquad B = \mu_o \frac{N}{\ell} i = \left(4\pi x10^{-7}\right)\left(\frac{1000}{1.00}\right)(0.100) = \underline{\underline{1.26x10^{-4}\,T}}$$

$$B = \mu_o \frac{N}{\ell} i \quad \text{The Magnetic Field Inside a Long Solenoid}$$

Example 6: The tube is bent into a circle to form a toroid. Find the magnetic field at the center.

The current enclosed by the indicated path is $i_{enclosed} = Ni$ where N is the total number of turns and i is the current in the wire. The magnetic field can only be a function of r and so it must be constant along the path shown. So, $\oint \vec{B}\cdot d\vec{s} = B\,2\pi r$. Using Ampere's Law,



$$\oint \vec{B}\cdot d\vec{s} = \mu_o i_{enclosed} \qquad B2\pi r = \mu_o Ni \qquad B = \mu_o \frac{N}{2\pi r} i.$$

Assume the inner circumference is equal to the length of the tube,

$r = r_i + \frac{D}{2} \qquad 2\pi r = 2\pi r_i + \pi D = \ell + \pi D$.

The field is, $B = \mu_o \frac{N}{\ell + \pi D} i = \left(4\pi x10^{-7}\right)\left(\frac{1000}{1.00 + \pi(0.0500)}\right)(0.100) = \underline{\underline{1.09x10^{-4}\,T}}$

Chapter 30 - Summary

Biot - Savart Rule $\vec{B} = \frac{\mu_o}{4\pi}\int \frac{Id\vec{s} \times \hat{r}}{r^2}$

The Magnetic Field Due to a Long Straight Wire $B = \frac{\mu_o I}{2\pi r}$

Ampere's Law $\oint \vec{B} \cdot d\vec{s} = \mu_o i_{enclosed}$

The Magnetic Field Inside a Long Solenoid $B = \mu_o \frac{N}{\ell} i$