

## Chapter 31 - Induction and Inductance

Problem Set #10 - due:

Ch 31 - 1, 5, 11, 28, 35, 43, 51, 59, 70, 74, 83, 100

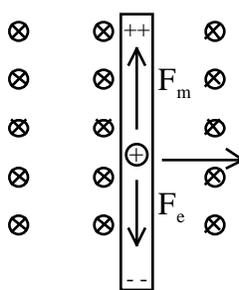
Lecture Outline

1. Faraday's Law of Induction
2. Motors and Generators
3. The Meissner Effect and Superconductivity
4. The Definition of Mutual Inductance
5. The Definition of Self Inductance
6. The LR Circuit
7. Energy Storage in Inductors and B-fields

Magnet, wire, coil, & galvanometer

(moving the coil vs. moving the magnet - it can't matter!)

### 1. Faraday's Law of Induction



Moving a wire through the field causes the charges within the wire to feel an upward magnetic force,  $\vec{F} = q \vec{v} \times \vec{B}$   $F = q v B$ . Positive charges accumulate at the top of the wire and negative charges at the bottom. This creates a downward electric field in the wire. The net force on the charges is given by the Second Law,  $F = ma$   $q B - qE = ma$ . Charges move upward until  $a=0$ . The electric field is then,  $E = v B$ .



If the wire was rolling along some rails connected to a voltmeter, the meter would give a reading due to the electric field in the wire. This voltage can be found from the E-field,

$$V = - \vec{E} \cdot d\vec{s} \quad V = E\ell \quad V = B\ell v$$

Moving through a magnetic field creates a potential difference just like a battery would.

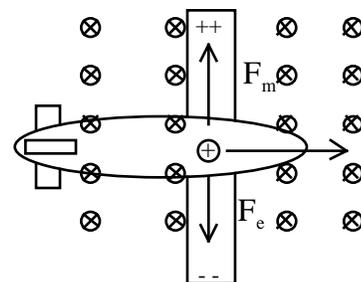
**Example 1:** Estimate the induced voltage across the 40.0m wingspread of an airplane traveling 800km/h (222m/s) perpendicular to the earth's magnetic field of 50.0μT.

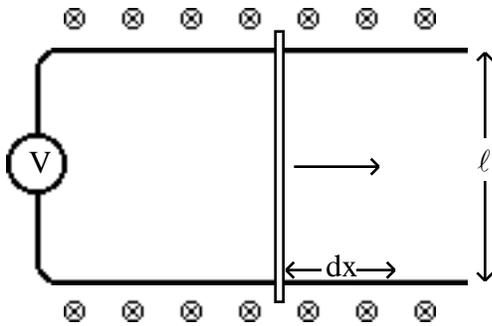
Using the Second Law when the charges in the wing have reached equilibrium,  $F = ma$   $q B - qE = 0$   $E = v B$ .

The induced voltage can be found from the E-field,

$$V = - \vec{E} \cdot d\vec{s} \quad V = E\ell \quad V = B\ell v$$

$$V = B\ell v \quad (50.0\mu T)(40.0m)(222m/s) \quad \underline{\underline{0.444V}}$$





The other way to look at this is to think about it in terms of the changing magnetic flux in the loop. The potential difference between the ends is,  $V = B\ell$ . Using the definition of velocity,  $V = B\ell \frac{dx}{dt} = B \frac{d}{dt}(\ell x) = B \frac{dA}{dt} = \frac{d}{dt}(BA) = \frac{d}{dt} \Phi$ .

This explains why moving the magnet creates a voltage just like moving the wire does. If the wire has  $N$  turns then the voltage is  $N$  times bigger. Notice that the current flows in a direction such as to fight the change in the field. This is called "Lenz's Rule." Putting all this together gives Faraday's Law,

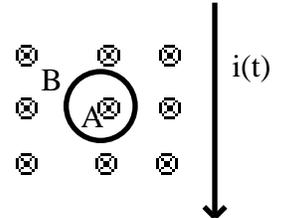
$$\mathcal{E} = -N \frac{d\Phi}{dt} \quad \text{Faraday's Law}$$

where we have defined the magnetic flux as,

$$\Phi_B = \int \vec{B} \cdot d\vec{A} \quad \text{The Definition of Magnetic Flux}$$

Beyond the Mechanical Universe (vol. 37 Ch 11,12,15,18,21,28)

*Example 2: A small loop of  $N$  turns and area  $A$  is in the same plane as a long straight wire carrying a current  $i = i_0 \sin \omega t$ . Find the induced voltage in the loop as a function of time and show that the peak-to-peak voltage is proportional to the peak magnetic field.*



From the previous chapter we know the field due to a long straight wire is,  $B = \frac{\mu_0 i}{2r}$ , where  $r$  is the distance from the wire to the center of the loop.

The loop is small enough that at any given time the field is approximately constant. Therefore, the magnetic flux through the small loop is,  $\Phi_B = \int \vec{B} \cdot d\vec{A} = B dA = B \int dA = BA = \frac{\mu_0 i}{2r} A$ .

Applying Faraday's Law,

$$\mathcal{E} = -N \frac{d}{dt} \left( \frac{\mu_0 i}{2r} A \right) = -N \frac{\mu_0}{2r} A \frac{di}{dt} = -\frac{\mu_0 NA}{2r} \frac{d}{dt} (i_0 \sin \omega t) = -\frac{\mu_0 NA}{2r} i_0 \cos \omega t = -\frac{\mu_0 i_0}{2r} NA \cos \omega t$$

The peak-to-peak voltage must be twice the maximum when  $\cos \omega t = 1$ ,

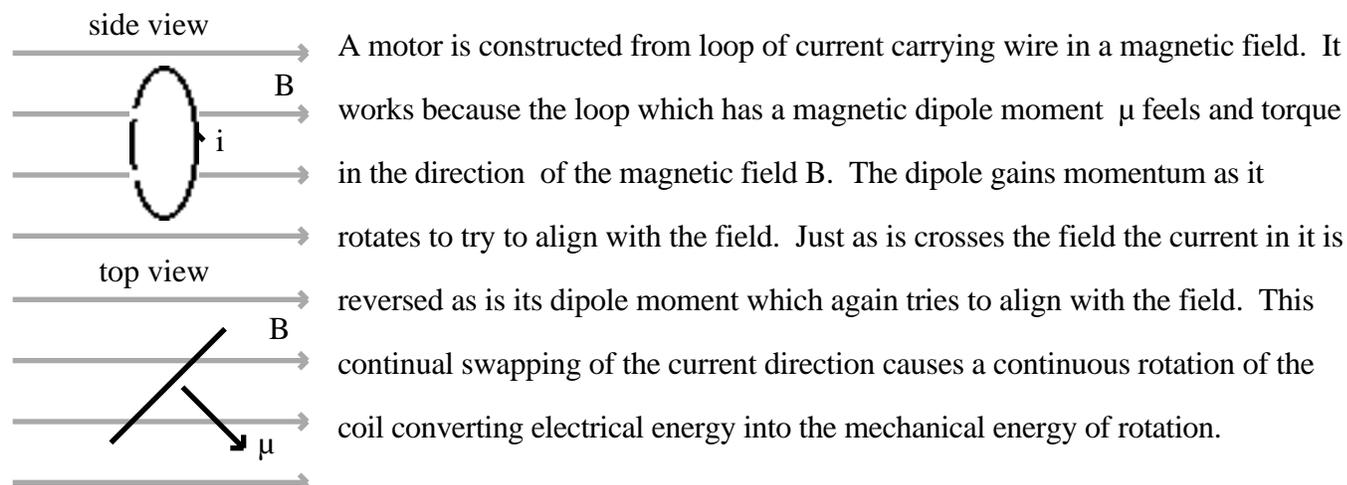
$$\mathcal{E}_{pp} = 2 \frac{\mu_0 i_0}{2r} NA = 2B_0 NA \quad \text{where } B_0 = \frac{\mu_0 i_0}{2r} \text{ is the peak field due to the wire.}$$

This is how the magnetic field probe you used in lab works! You can now see that the peak-to-peak voltage is proportional to the field assuming the frequency is kept constant.

## 2. Motors and Generators

A motor and a generator are physically the same device. A motor converts electrical energy into mechanical energy while a generator converts mechanical energy into electrical energy.

### Motors:



Demonstrate a motor

### Generators:

A generator is just a motor running backwards. Mechanical energy is used to turn the coil. This energy can come from water falling over a dam for instance. The changing magnetic flux induces a voltage and therefore a current in the coil according to Faraday's Law. The generator converts the mechanical energy into electrical energy.

Demonstrate with a generator and galvanometer

**Example 3:** A coil of  $N$  turns and area  $A$  is rotated at a frequency  $f$  about an axis perpendicular to a magnetic field  $B$ . Find (a) the induced voltage as a function of time, (b) the peak voltage and (c) the rms voltage.

(a) When the coil makes an angle with the magnetic field the flux through the coils is,

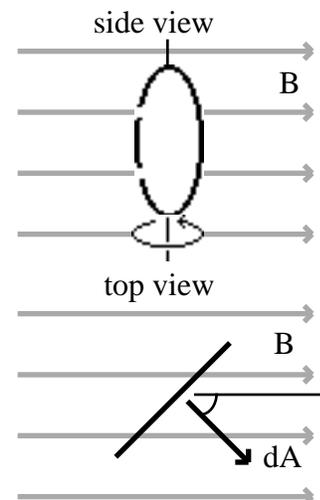
$$\Phi_B = \int \vec{B} \cdot d\vec{A} = B \cos \theta \, dA = B \cos \theta \int dA = BA \cos \theta$$

If  $\theta = 0$  at  $t=0$  then  $\theta = 2\pi ft$  and  $\Phi_B = BA \cos 2\pi ft$ . The induced voltage can be found from Faraday's Law,

$$\mathcal{E} = -N \frac{d\Phi_B}{dt} = \underline{\underline{NBA 2\pi f \sin 2\pi ft}}$$

(b) The peak voltage occurs when  $\sin 2\pi ft = 1$  and its value is  $\mathcal{E}_0 = \underline{\underline{NBA 2\pi f}}$ .

(c) rms means root mean square or the square root of the average value of the square of the voltage. Using the definition of average,

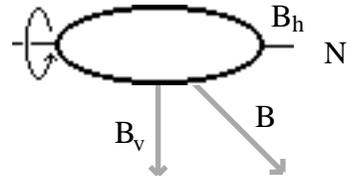


$$\frac{1}{2} \int_0^T i^2 dt = \frac{1}{2} \int_0^T \sin^2 2\pi f t dt = \frac{1}{2} \int_0^{2\pi} \sin^2 u du = \frac{1}{2} \int_0^{2\pi} \frac{1 - \cos 2u}{2} du = \frac{1}{4} \int_0^{2\pi} (1 - \cos 2u) du = \frac{1}{4} [u - \frac{1}{2} \sin 2u]_0^{2\pi} = \frac{1}{4} (2\pi) = \frac{\pi}{2}$$

$$i_{\text{rms}} = \sqrt{\frac{\pi}{2}} = \frac{1}{\sqrt{2}} i_0 = \frac{1}{\sqrt{2}} N B A 2\pi f r^2$$

**Example 4:** A coil of 1000 turns and 12.0cm radius flips 180° about an axis that points northward. The coil has a resistance of 4.80 Ω. The vertical component of Earth's magnetic field is 46.0μT. Find the total charge that flows when the coils flips.

Notice that the horizontal component of Earth's field contributes no flux. All the flux through the loop is due to the vertical component.



Faraday's Law requires,  $\mathcal{E} = -N \frac{d\Phi_B}{dt}$ .

The charge that flows is related to Ohm's Rule,  $\mathcal{E} = IR = \frac{dq}{dt} R$ .

Setting the voltages equal,

$$\frac{dq}{dt} R = -N \frac{d\Phi_B}{dt} \quad R dq = -N d\Phi_B \quad \int_0^Q R dq = -N \int_{\Phi_i}^{\Phi_f} d\Phi_B \quad RQ = -N(\Phi_f - \Phi_i)$$

The initial flux is just the product of the vertical component of the field and the area. The final flux is just the opposite of the initial flux,  $RQ = -N(-B_v \pi r^2 - B_v \pi r^2) = 2NB_v \pi r^2$ .

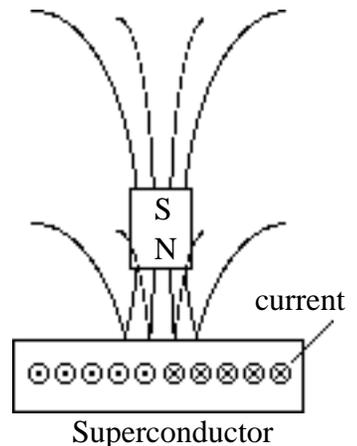
$$Q = \frac{2NB_v \pi r^2}{R} = \frac{2(1000)(46.0 \times 10^{-6}) \pi (0.120)^2}{4.80} = \underline{\underline{8.67 \times 10^{-4} \text{ C}}}$$

### 3. The Meissner Effect and Superconductivity

picture, floating magnet toy, coil & power supply, high-TC superconductor

Why does a floating magnet imply superconductivity?

Two magnets repel each other, but the lower magnet can be replaced with a current carrying coil. The floating magnet implies a current in the superconductor. Since there is no power source, and the current lasts indefinitely, the material is conducting without resistance and is therefore a superconductor.



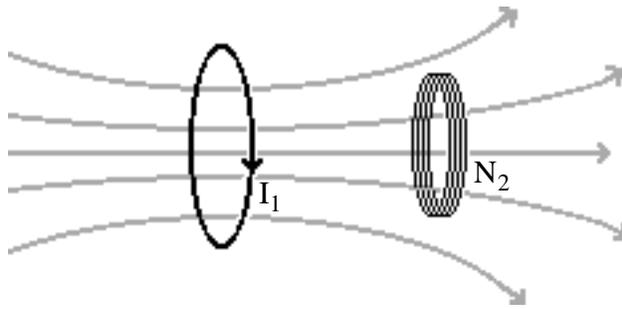
How does the current get there to begin with?

If you bring the magnet toward the superconductor, Faraday's Law explains the induced current just like moving the magnet around near a coil. In ordinary conductors these currents die away due to the resistance. The currents in the superconductor continue.

The Meissner Effect

Set the magnet on a warm chunk of superconducting material and nothing happens because the material is only superconducting at low temperatures. When you cool the sample, the magnet lifts off and hovers above the superconductor. This is not explained by Faraday's Law, but it is called "The Meissner Effect." It is caused by the fact that superconductors exclude magnetic fields just like ordinary conductors exclude electric fields. In order for the superconductor to keep the field inside zero, it must set up a current that cancels out the field due to the magnet. These opposing fields cause the magnet to hover.

#### 4. The Definition of Mutual Inductance



If a coil carrying a current  $I_1$  is near a coil with  $N_2$  turns, the magnetic field caused by the first coil will create a flux through the second coil. If the current in the first coil changes, the flux through the second coil will change and according to Faraday's Law a voltage will be induced in the second coil. The relationship between the voltage induced in the second coil and the

rate of change of current in the first coil is called the "Mutual Inductance."

To find voltage induced in the second coil use Faraday's Law,  $\mathcal{E}_2 = -N_2 \frac{d\Phi_{21}}{dt}$ .

The flux through coil 2 caused by coil 1 can be found from the definition of flux,  $\Phi_{21} = \int_{\text{coil 2}} \vec{B}_1 \cdot d\vec{A}_2$ .

The field caused by coil 1 can be found by the Biot-Savart Rule,  $\vec{B}_1 = \frac{\mu_0 I_1}{4\pi} \int_{\text{coil 1}} \frac{d\vec{s} \times \hat{r}}{r^2}$ .

The flux can now be written,  $\Phi_{21} = \frac{\mu_0 I_1}{4\pi} \int_{\text{coil 2}} \int_{\text{coil 1}} \frac{d\vec{s} \times \hat{r}}{r^2} \cdot d\vec{A}_2 = \frac{I_1}{N_2} M_{21}$

where  $M_{21} = \frac{\mu_0 N_2}{4\pi} \int_{\text{coil 2}} \int_{\text{coil 1}} \frac{d\vec{s} \times \hat{r}}{r^2} \cdot d\vec{A}_2$ . Notice that  $M_{21}$  only depends on the shapes of the coils and the relative positions not on the current in coil 1 or on time.

Putting the flux into Faraday's Law,  $\mathcal{E}_2 = -N_2 \frac{d}{dt} \left( \frac{I_1}{N_2} M_{21} \right) = -M_{21} \frac{dI_1}{dt}$ . This equation is usually called the definition of mutual inductance,

$$\mathcal{E}_2 = -M_{21} \frac{dI_1}{dt} \quad \text{The Definition of Mutual Inductance}$$

#### 5. The Definition of Self Inductance

Beyond the Mechanical Universe (vol. 37 Ch 25)

There is no reason that coil 1 and coil 2 have to be separate. They could be part of a solenoid for instance. In this case we call the quantity that relates the induced voltage to the rate of change of current the "Self Inductance" and we use the symbol  $L$  instead of  $M$ .

$$-\mathcal{E} = -L \frac{dI}{dt} \quad \text{The Definition of Self Inductance}$$

Example 5: A solenoid has 1000 turns of wire. It is 10.0cm long and 1.00cm in diameter. Estimate the self inductance.

Combining Faraday's Law and the definition of self inductance,

$$-L \frac{dI}{dt} = -N \frac{d\Phi}{dt} \quad LdI = Nd\Phi \quad LI = N\Phi \quad L = \frac{N\Phi}{I}$$

The flux can be found from its definition by using the field due to the solenoid,

$$\vec{B} \cdot d\vec{A} = \mu_0 \frac{N}{\ell} IdA = \mu_0 \frac{N}{\ell} I dA = \mu_0 \frac{N}{\ell} I \frac{D^2}{4}$$

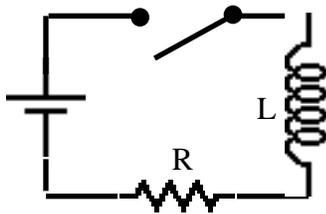
The self inductance is now,

$$L = \frac{N}{I} \mu_0 \frac{N}{\ell} I \frac{D^2}{4} = \frac{\mu_0 N^2 D^2}{4\ell} = \frac{(4 \times 10^{-7})(1000)^2 (0.0100)^2}{4(0.100)} = \underline{\underline{9.87 \times 10^{-4} \frac{T \cdot m^2}{A} = 987 \mu H}}$$

Note that the self inductance depends on geometry not on current. Also note the units are Henries.



### 6. The LR Circuit



When the switch is first closed the current is zero, but changing rapidly. Therefore, all of the voltage will be dropped on the inductor and none on the resistor. After a long time, the current will build up to the point where all the voltage is dropped on the resistor and therefore the current stops changing. In between we can use the loop theorem to find the current in the circuit at any

time,  $V + V_L + V_R = 0 \quad -L \frac{dI}{dt} - IR = 0$ . Separating the equation and setting up the integral,

$$\frac{dI}{dt} = \frac{-IR}{L} \quad \int_0^I \frac{dI}{-IR} = \int_0^t \frac{dt}{L}$$

The left hand side can be done by choosing

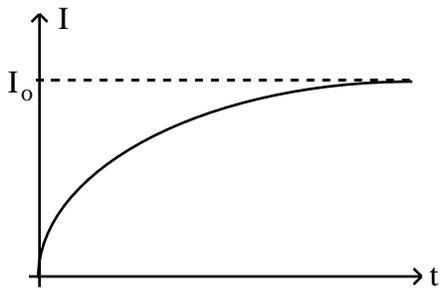
$$u = -IR \quad du = -RdI$$

Now we can do the integral,

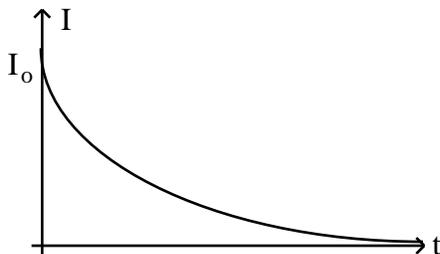
$$-\frac{1}{R} \int_{I=0}^I \frac{du}{u} = \frac{t}{L} \quad \ln(-IR)|_0^I = -\frac{R}{L} t \quad \ln\left(\frac{-IR}{-IR_0}\right) = -\frac{R}{L} t \quad I = \frac{R}{L} 1 - e^{-\frac{R}{L} t}$$

Since  $\frac{R}{L}$  is the peak current in the circuit, it is convenient to write the current as,

$$I = I_0 \left( 1 - e^{-\frac{R}{L} t} \right) \quad \text{"Charging" LR Circuit}$$



The current as a function of time for a "charging" LR circuit is shown at the left. The word charging is used, but keep in mind that the inductor is storing flux (or field, or energy) not charge like a capacitor would.



For a "discharging" LR circuit the initial current would be large enough for all the voltage to be dropped across the resistor. As the resistor removes energy from the circuit the current begins to drop toward zero. The loop theorem can again be used to find the current as a function of time. The result is,

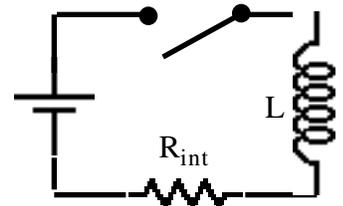
$$I = I_0 e^{-\frac{R}{L}t} \quad \text{"Discharging" LR Circuit}$$

The graph of current versus time is shown at the left.

**Example 6:** The inductor from example 5 is made of copper. Find (a) its resistance, (b) the equilibrium current when it is connected to a 1.50V battery and (c) the time required for it to reach 99% of this equilibrium current.

(a) Using the definition of resistance,  $R = \frac{\ell}{A} = \frac{DN}{a^2}$  where  $a$  is the radius of the wire. Assuming the coils are close packed  $2aN = \ell$  where  $\ell$  is the length of the solenoid. Solving,

$$R = \frac{DN}{\left(\frac{\ell}{2N}\right)^2} = \frac{4DN^3}{\ell^2} = \frac{4(1.7 \times 10^{-8})(0.0100)(1000)^3}{(0.100)^2} = \underline{\underline{68.0}} .$$



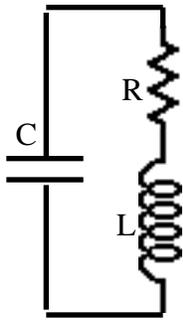
(b) After a while the current will build up. It can grow until all the voltage is dropped across the resistor. According to Ohm's Rule,  $V = I_0 R$   $I_0 = \frac{V}{R} = \frac{1.50V}{68.0} = \underline{\underline{22.1mA}}$ .

(c) Starting with the equation for a charging LR circuit  $I = I_0 \left(1 - e^{-\frac{R}{L}t}\right)$  and solving for  $t$ ,

$$\frac{I}{I_0} = 1 - e^{-\frac{R}{L}t} \quad e^{-\frac{R}{L}t} = 1 - \frac{I}{I_0} \quad -\frac{R}{L}t = \ln \left(1 - \frac{I}{I_0}\right) \quad t = -\frac{L}{R} \ln \left(1 - \frac{I}{I_0}\right) .$$

Putting in the numbers,  $t = -\frac{9.87 \times 10^{-4}}{68.0} \ln(1 - 0.99) = \underline{\underline{66.8\mu s}}$

### 7. Energy Storage in Inductors and B-fields



The energy stored in the inductor can be found by starting with the loop theorem,

$$V_L + V_R + V_C = 0 \quad L \frac{dI}{dt} + IR + \frac{Q}{C} = 0.$$

Multiplying through by the current,  $LI \frac{dI}{dt} + I^2 R + I \frac{Q}{C} = 0$ . The middle term is the power consumed by the resistor. It's a good guess to assume the other terms are the power flow into the capacitor and the inductor. Since we know the answer for the capacitor let's start there,

$$P_C = I \frac{Q}{C} \quad \frac{dU_C}{dt} = \frac{dQ}{dt} \frac{Q}{C} \quad \int_0^{U_C} dU_C = \int_0^Q \frac{1}{C} Q dQ \quad U_C = \frac{1}{2} \frac{Q^2}{C}.$$

Since this is the right answer, we should play the same game with the inductance term,

$$P_L = LI \frac{dI}{dt} \quad \frac{dU_L}{dt} = LI \frac{dI}{dt} \quad \int_0^{U_L} dU_L = \int_0^I LI dI \quad U_L = \frac{1}{2} LI^2.$$

$$U_L = \frac{1}{2} LI^2 \quad \text{Energy Storage in an Inductor}$$

*Example 7:* For the inductor circuit of example 6 ( $L=987\mu\text{H}$ ,  $R=68.0 \Omega$ ,  $\mathcal{E}=1.50\text{V}$ ) find the energy stored when the current reaches equilibrium.

From example 2 the equilibrium current is 22.1mA. Using the energy stored in the inductor,

$$U_L = \frac{1}{2} LI^2 = \frac{1}{2} (987 \times 10^{-6}) (0.0221)^2 = \underline{\underline{0.241 \mu\text{J}}}.$$

If we associate this energy with the magnetic field created by the inductor instead of associating it with the inductor itself, we can find the energy density in magnetic fields,  $u_m = \frac{U}{\text{vol}} = \frac{\frac{1}{2} LI^2}{\frac{1}{4} D^2 \ell}$ . Using the inductance of a solenoid found in example 1,

$$L = \frac{\mu_0 N^2 D^2}{4\ell} \quad u_m = \frac{\frac{1}{2} \frac{\mu_0 N^2 D^2}{4\ell} I^2}{\frac{1}{4} D^2 \ell} = \frac{\frac{1}{2} \mu_0 N^2 I^2}{\ell^2} = \frac{1}{2\mu_0} \frac{\mu_0 NI}{\ell}^2.$$

The stuff in parentheses is the magnetic field due to the solenoid so,

$$u_m = \frac{B^2}{2\mu_0} \quad \text{Energy Density in a Magnetic Field}$$

Recall the energy density in the electric field is given by the similar expression,  $u_e = \frac{1}{2} \epsilon_0 E^2$ .

*Example 8:* In some region of space the electric field and the magnetic field give equal contributions to the energy density. Find the ratio of E to B.

$$\text{Given, } u_m = u_e \quad \frac{B^2}{2\mu_0} = \frac{1}{2} \epsilon_0 E^2 \quad \frac{E^2}{B^2} = \frac{1}{\mu_0 \epsilon_0} \quad \frac{E}{B} = \frac{1}{\sqrt{\mu_0 \epsilon_0}}.$$

$$\text{Putting in the numbers, } \frac{1}{\sqrt{\mu_0 \epsilon_0}} = \frac{1}{\sqrt{(4 \times 10^{-7})(8.85 \times 10^{-12})}} = \underline{\underline{3.00 \times 10^8 \text{ m/s}}}.$$

The speed of light! Hmmm.

**Chapter 31 - Summary**

Faraday's Law  $= -N \frac{d\Phi_B}{dt}$

The Definition of Magnetic Flux  $\Phi_B = \int \vec{B} \cdot d\vec{A}$

The Definition of Mutual Inductance  $\mathcal{M}_{21} = -M_{21} \frac{dI_1}{dt}$

The Definition of Self Inductance  $\mathcal{E} = -L \frac{dI}{dt}$

The LR Circuit: "charging"  $I = I_0 (1 - e^{-\frac{R}{L}t})$       "discharging"  $I = I_0 e^{-\frac{R}{L}t}$

Energy Storage in an Inductor  $U_L = \frac{1}{2} LI^2$

Energy Density in a Magnetic Field  $u_m = \frac{B^2}{2\mu_0}$