Problem 4.39

(a)Start with the radial equation 4.35.

(b)Substitute $R = \frac{u}{r}$ to get $r\frac{d^2u}{dr^2} - \frac{2mr^2}{\hbar^2}\left[\frac{1}{2}m\omega^2r^2 - E\right]\frac{u}{r} = \ell(\ell+1)\frac{u}{r}$.

(c)Use the dimensionless variables $\xi = \sqrt{\frac{m\omega}{\hbar}}r$ and $\varepsilon = \frac{E}{\frac{1}{2}\hbar\omega}$ to get,

$$\boxed{-\frac{d^2u}{d\xi^2} + \left[\xi^2 + \frac{\ell(\ell+1)}{\xi^2} - \varepsilon\right]u = 0}.$$

(d)Show that for large ξ, $u \approx e^{-\frac{\xi^2}{2}}$ consistent with equation 2.72.

(e)Show that for small ξ, $u \approx \xi^{\ell+1}$ consistent with equation 4.59.

(f)Define $u(\xi) \equiv \xi^{\ell+1}e^{-\frac{\xi^2}{2}}v(\xi)$ and show the radial equation becomes,

$$\frac{d^2v}{d\xi^2} + 2\left(\frac{\ell+1}{\xi} - \xi\right)\frac{dv}{d\xi} + (\varepsilon - 2\ell - 3)\xi = 0.$$

(g)Now use the power series, $v(\xi) = \sum_{n=0}^{\infty} a_n\xi^n$ and require that the coefficients of each power of ξ go to zero.

(h)Show that for odd n, $a_n = 0$ and for even n the recursion relation is,

$$a_{n+2} = a_n\frac{2\ell + 2n + 3 - \varepsilon}{(n+1)(n+2) + 2(\ell+1)(n+2)}.$$

(i)Make physical arguments to require $E = \frac{1}{2}\hbar\omega(2\ell + 2n_{\max} + 3)$.

(j)Show the resulting energies and degeneracies match the results we got in Cartesian coordinates in problem 4.38