Problem 4.6

(a)Use equation 4.28 to express the Associated Legrendre Polynomials in terms of derivatives of $\left(x^2-1\right)$.

(b)Choose $\ell > \ell'$ and perform integration by parts to increase the number of derivatives of $(x^2-1)^{\ell'}$ by one.

(c)Explain carefully why the integrated term must be zero due to too few derivatives of $\left(x^2-1\right)$.

(d)Complete ℓ integrations by parts explaining why the integrated term will always go to zero.

(e)Explain carefully why the integrand must be zero unless $\ell = \ell'$. You have now shown the Associated Legrendre Polynomials are orthogonal.

(f)When $\ell = \ell'$ show the integrand becomes $(2\ell)!\left(x^2-1\right)^{\ell}$. Now you should have,

$$\int_{-1}^{1} P_\ell(x) P_{\ell'}(x)\, dx = \frac{(-1)^\ell (2\ell)!}{2^{2\ell}(\ell!)^2}\delta_{\ell,\ell'}\int_{-1}^{1}\left(x^2-1\right)^\ell dx .$$

(g)Show that you can rewrite, $\int_{-1}^{1}\left(x^2-1\right)^\ell dx = (-1)^\ell \int_0^\pi \sin^{2\ell+1}\theta\, d\theta$.

(h)Use integration by parts to show, $\int_0^\pi \sin^{2\ell+1}\theta\, d\theta = \frac{2\ell}{2\ell+1}\int_0^\pi \sin^{2\ell-1}\theta\, d\theta$.

(i)Integrate ℓ times to show

$$\int_0^\pi \sin^{2\ell+1}\theta\, d\theta = \frac{(2\ell)(2\ell-2)(2\ell-4)\cdots(6)(4)(2)}{(2\ell+1)(2\ell-1)(2\ell-3)\cdots(5)(3)(1)}\int_0^\pi \sin\theta\, d\theta .$$

(j)Simplify the result to show, $\int_0^\pi \sin^{2\ell+1}\theta\, d\theta = 2\frac{\left(2^\ell \ell!\right)^2}{(2\ell+1)!}$.

(k)Finally combine this result with the results from parts (f) and (g).