Gauge Invariance Issues and the Aharanov-Bohm Effect

Replacing $\vec{p} \to \vec{p} - q\vec{A}$ in Schrodinger's Equation has some interesting consequences due to the vector potential for a given magnetic field is not being unique. Changing from one vector potential to another is called a "Gauge Transformation." Gauge transformations have a distinct effect on the wave function. The Aharonov-Bohm Effect illustrates some of these features. These results indicate a need to rethink our notion that charged particles are only affected by local electric and magnetic fields.

(a)Write the magnetic field $\vec{B}$ in terms of the vector potential $\vec{A}$. Show that the replacement, $\vec{A} \to \vec{A} + \nabla f$ where f is any function of the coordinates and time, leaves the fields unchanged.

(b)Write the time independent Schrodinger's Equation replacing $\vec{p} \to \vec{p} - q\vec{A}$. Call the solution, ψ.

(c)Write Schrodinger's Equation again replacing $\vec{A} \to \vec{A} + \nabla f$. Call the solution, ψ'. Show that $\psi' = \psi e^{i\Lambda(r)}$ satisfies Schrodinger's Equation for one particular value of Λ. Find this Λ in terms of f and explain the meaning of the result. Here are some hints,

1. Define the operator $\vec{P} \equiv \vec{p} - q\vec{A} - q\nabla f$ and show that $\vec{P}\psi' = e^{i\Lambda}\left(\hbar\nabla\Lambda + \vec{P}\right)\psi$.
2. Show that Schrodinger's Equation becomes $\boxed{\frac{1}{2m}\left(\vec{p} - q\vec{A} - q\nabla f + \hbar\nabla\Lambda\right)^2 \psi + V\psi = E\psi}$ which requires that the last two terms in the parentheses sum to zero.

(d)Consider an infinitely long solenoid of radius R_o aligned along the z-axis that has a constant magnetic field B inside. Show that the vector potential inside $\vec{A} = \frac{B}{2}(\hat{k} \times \vec{r})$ and the vector potential outside $\vec{A} = \frac{B}{2}(\hat{k} \times \vec{r})\frac{R_o^2}{r^2}$ are correct by using them to find the magnetic fields inside and out. Note that $\vec{r} = x\hat{i} + y\hat{j}$ and has no z-component.

(e)Consider an electron outside a long solenoid where the magnetic field is zero. Show that the vector potential in such a region can be written solely as the gradient of a scalar function, f. Solve for f in terms of a line integral of this vector potential over a path s. Use the result of part (c) to find the phase shift in the wave function for an electron over the path s.

(f)Find the phase difference of an electron taken around a closed path that surrounds, but doesn't enter solenoid. Use Stokes theorem to write the line integral in terms of a surface integral, then complete the integration.

(g)Consider the experiment at the right. Find the phase difference between electrons traveling along the two paths shown. Note that the two paths would form a closed loop except that s_1 and s_2 both head from the source to the screen, which is the cause of the phase difference. Describe the effect of the B field on the interference pattern.