**Physics 235B – Simplified Quantum Electrodynamics**

1. Begin with a classical harmonic oscillator of mass m and natural frequency ω. Show that the Hamiltonian is given by,
$$H = \tfrac{1}{2m} p^2 + \tfrac{1}{2} m\omega^2 x^2$$
where the momentum and position are related by,
$$\frac{dp}{dt} = -m\omega^2 x \text{ and } \frac{dx}{dt} = \frac{1}{m} p.$$

2. Write down the energy eigenvalues, the expressions for the raising and lowering operators, and the number operator for the quantum solution to the harmonic oscillator.

3. Use Maxwell's Equations for free space to find the wave equation for the electric field and magnetic fields.

4. For a box of volume V, show that the one-dimensional standing waves below satisfy the wave equation.
$$E_y(x,t) = \sqrt{\frac{2\eta\omega^2}{\varepsilon_o V}}\mathcal{E}(t)\sin(\tfrac{\omega x}{c}) \text{ and } B_z(x,t) = \sqrt{\frac{2\mu_o}{\eta V}}\beta(t)\cos(\tfrac{\omega x}{c})$$
where η is an arbitrary factor determining the size of the field. Find the conditions on $\mathcal{E}$(t) and β(t).

5. Use Maxwell's Equations to show that,
$$\frac{d\beta}{dt} = -\eta\omega^2\mathcal{E} \text{ and } \frac{d\mathcal{E}}{dt} = \frac{1}{\eta}\beta$$
and show that the conditions from problem 4 are satisfied.

6. Write the total energy contained in the fields in terms of ε and β. Note that the integration is over an integer number of wavelengths required by the boundary condition in the box.

7. Comparing the results of problems 5 and 6 with problems 1 and 2, write down the energy eigenvalues, the expressions for the raising and lowering operators, and the number operator for the quantized electromagnetic field.

1. The Hamiltonian is the sum of the kinetic and potential energies,

$$H = \tfrac{1}{2m} p^2 + \tfrac{1}{2} m\omega^2 x^2.$$

Using the Second Law, $F = -kx \Rightarrow \dfrac{dp}{dt} = -m\omega^2 x$.

The definition of momentum states, $\dfrac{dx}{dt} = \dfrac{1}{m} p$.

2. $E_n = (n + \tfrac{1}{2})\hbar\omega$, $\hat{a}_\pm = \dfrac{1}{\sqrt{2\hbar\omega m}}\left(\mp i\hat{p} + m\omega\hat{x}\right)$ and $\hat{N} = \hat{a}_+\hat{a}_-$

3. Given $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \Rightarrow \nabla \times \left(\nabla \times \vec{E}\right) = -\dfrac{\partial}{\partial t}\nabla \times \vec{B} \Rightarrow -\nabla^2 \vec{E} + \nabla(\nabla \bullet \vec{E}) = -\dfrac{\partial}{\partial t}\nabla \times \vec{B}$

Using $\nabla \times \vec{B} = \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t}$ and $\nabla \bullet \vec{E} = 0$ gives the wave equation $\nabla^2 \vec{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \vec{E}}{\partial t^2}$.

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4. $\nabla^2 \vec{E} = \dfrac{1}{c^2}\dfrac{\partial^2 \vec{E}}{\partial t^2} \Rightarrow -\dfrac{\omega^2}{c^2}\sqrt{\dfrac{2\eta\omega^2}{\varepsilon_o V}}\mathcal{E}(t)\sin(\tfrac{\omega x}{c}) = \dfrac{1}{c^2}\sqrt{\dfrac{2\eta\omega^2}{\varepsilon_o V}}\dfrac{\partial^2 \mathcal{E}(t)}{\partial t^2}\sin(\tfrac{\omega x}{c})$

The condition on $\mathcal{E}$(t) is, $\dfrac{d^2\mathcal{E}}{dt^2} = -\omega^2\mathcal{E}$.

$\nabla^2 \vec{B} = \dfrac{1}{c^2}\dfrac{\partial^2 \vec{B}}{\partial t^2} \Rightarrow -\dfrac{\omega^2}{c^2}\sqrt{\dfrac{2\mu_o}{\eta V}}\beta(t)\cos(\tfrac{\omega x}{c}) = \dfrac{1}{c^2}\sqrt{\dfrac{2\mu_o}{\eta V}}\dfrac{\partial^2 \beta(t)}{\partial t^2}\cos(\tfrac{\omega x}{c})$

The condition on β(t) is, $\dfrac{d^2\beta}{dt^2} = -\omega^2\beta$.

5. Using $\nabla \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t} \Rightarrow \dfrac{\partial E_y}{\partial x} = -\dfrac{\partial B_z}{\partial t} \Rightarrow \dfrac{\omega}{c}\sqrt{\dfrac{2\eta\omega^2}{\varepsilon_o V}}\mathcal{E}(t)\cos(\tfrac{\omega x}{c}) = -\sqrt{\dfrac{2\mu_o}{\eta V}}\dfrac{\partial \beta(t)}{\partial t}\cos(\tfrac{\omega x}{c})$

$\dfrac{\omega}{c}\sqrt{\dfrac{\eta\omega^2}{\varepsilon_o}}\mathcal{E}(t) = -\sqrt{\dfrac{\mu_o}{\eta}}\dfrac{\partial \beta(t)}{\partial t} \Rightarrow \dfrac{d\beta}{dt} = -\eta\omega^2\mathcal{E}$

$\nabla \times \vec{B} = \dfrac{1}{c^2}\dfrac{\partial \vec{E}}{\partial t} \Rightarrow \dfrac{\partial B_z}{\partial x} = -\dfrac{1}{c^2}\dfrac{\partial E_y}{\partial t} \Rightarrow -\dfrac{\omega}{c}\sqrt{\dfrac{2\mu_o}{\eta V}}\beta(t)\sin(\tfrac{\omega x}{c}) = -\dfrac{1}{c^2}\sqrt{\dfrac{2\eta\omega^2}{\varepsilon_o V}}\dfrac{\partial \mathcal{E}(t)}{\partial t}\sin(\tfrac{\omega x}{c})$

$\omega\sqrt{\dfrac{\mu_o}{\eta}}\beta(t) = c\sqrt{\dfrac{\eta\omega^2}{\varepsilon_o}}\dfrac{\partial \mathcal{E}(t)}{\partial t} \Rightarrow \dfrac{d\mathcal{E}}{dt} = \dfrac{1}{\eta}\beta$

Taking the time derivative of the first equation and substituting from the second equation,
$$\frac{d^2\beta}{dt^2} = -\eta\omega^2 \frac{d\mathcal{E}}{dt} = -\eta\omega^2 \frac{1}{\eta}\beta \Rightarrow \frac{d^2\beta}{dt^2} = -\omega^2\beta.$$

Vice-versa yields, $\frac{d^2\mathcal{E}}{dt^2} = \frac{1}{\eta}\frac{d\beta}{dt} = -\frac{1}{\eta}\eta\omega^2\mathcal{E} \Longrightarrow \frac{d^2\mathcal{E}}{dt^2} = -\omega^2\mathcal{E}$.

6. Write the total energy contained in the fields in terms of ε and β. Note that the integration is over an integer number of wavelengths required by the boundary condition in the box.

The energy density must be integrated over the box,
$$H = \int_V \left(\tfrac{1}{2\mu_o} B^2 + \tfrac{1}{2}\varepsilon_o E^2\right) dxdydz$$
$$H = \int_V \left(\tfrac{1}{2\mu_o} \frac{2\mu_o}{\eta V}\beta^2 \cos^2(\tfrac{\omega x}{c}) + \tfrac{1}{2}\varepsilon_o \frac{2\eta\omega^2}{\varepsilon_o V}\mathcal{E}^2 \sin^2(\tfrac{\omega x}{c})\right) dxdydz$$
$$H = \tfrac{1}{V}\int_V \left(\frac{1}{\eta}\beta^2 \cos^2(\tfrac{\omega x}{c}) + \eta\omega^2\mathcal{E}^2 \sin^2(\tfrac{\omega x}{c})\right) dxdydz$$
$$H = \tfrac{1}{V}\left(\frac{1}{\eta}\beta^2 \int_0^L \cos^2(\tfrac{\omega x}{c})dx + \eta\omega^2\mathcal{E}^2 \int_0^L \sin^2(\tfrac{\omega x}{c})dx\right)\int dy \int dz$$
Since the length of the box is an integral number of wavelengths, $L = n\lambda = \frac{nc}{f} = 2\pi n\frac{c}{\omega}$.
$$H = \tfrac{1}{V}\left(\frac{1}{\eta}\beta^2 \frac{c}{\omega}\left[\tfrac{\omega x}{2c} + \tfrac{1}{4}\sin(\tfrac{2\omega x}{c})\right] + \eta\omega^2\mathcal{E}^2 \frac{c}{\omega}\left[\tfrac{\omega x}{2c} - \tfrac{1}{4}\sin(\tfrac{2\omega x}{c})\right]\right)_0^{2\pi n\frac{c}{\omega}} \int dy \int dz$$
$$H = \tfrac{1}{V}\left(\frac{1}{\eta}\beta^2 \frac{c}{\omega}\left[\tfrac{\omega}{2c}L\right] + \eta\omega^2\mathcal{E}^2 \frac{c}{\omega}\left[\tfrac{\omega}{2c}L\right]\right)\int dy \int dz = \tfrac{1}{V}\left(\frac{1}{\eta}\beta^2 + \eta\omega^2\mathcal{E}^2\right)\frac{L}{2}\int dy \int dz$$

The remaining integrals along with the L from the x integration cancel with the volume of the box,
$$H = \frac{1}{2\eta}\beta^2 + \frac{1}{2}\eta\omega^2\mathcal{E}^2$$

Comparing the results of problems 4 and 5 with problems 1 and 2, write down the energy eigenvalues, the expressions for the raising and lowering operators, and the number operator for the quantized electromagnetic field.

| Harmonic Oscillator | Electromagnetic Field |
|---|---|
| $H = \frac{1}{2m} p^2 + \frac{1}{2} m\omega^2 x^2$ | $H = \frac{1}{2\eta}\beta^2 + \frac{1}{2}\eta\omega^2\mathcal{E}^2$ |
| $\frac{dp}{dt} = -m\omega^2 x$ | $\frac{d\beta}{dt} = -\eta\omega^2\mathcal{E}$ |
| $\frac{dx}{dt} = \frac{1}{m} p$ | $\frac{d\mathcal{E}}{dt} = \frac{1}{\eta}\beta$ |
| $E_n = (n + \frac{1}{2})\hbar\omega$ | $E_n = (n + \frac{1}{2})\hbar\omega$ |
| $\hat{a}_\pm = \frac{1}{\sqrt{2\hbar\omega m}}\left(\mp i\hat{p} + m\omega\hat{x}\right)$ | $\hat{a}_\pm = \frac{1}{\sqrt{2\hbar\omega\eta}}\left(\mp i\hat{\beta} + \eta\omega\hat{\mathcal{E}}\right)$ |
| $\hat{N} = \hat{a}_+\hat{a}_-$ | $\hat{N} = \hat{a}_+\hat{a}_-$ |

Most books choose dimensionless raising and lowering operators (Sakuri for example).